

ON NONLOCAL PARABOLIC STEADY-STATE EQUATIONS OF COOPERATIVE OR COMPETING SYSTEMS

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ABSTRACT. Some systems of parabolic equations with nonlocal initial conditions are studied. The systems arise when considering steady-state solutions to diffusive age-structured cooperative or competing species. Local and global bifurcation techniques are employed to derive a detailed description of the structure of positive coexistence solutions.

1. INTRODUCTION AND MAIN RESULTS

In this paper we characterize the structure of positive solutions to certain systems of coupled parabolic equations with nonlocal initial conditions. Such systems arise as steady-state equations of two age-structured diffusive populations which inhabit the same spatial region. The interaction between the two species is either of cooperative, competing, or predator-prey type leading to different structures of positive solutions. Denoting the density of the two species by $u = u(a, x) \geq 0$ and $v = v(a, x) \geq 0$ with $a \in (0, a_m)$ and $x \in \Omega \subset \mathbb{R}^n$ referring to age and spatial position, respectively, the models we shall focus on are of the form

$$\partial_a u - \Delta_D u = -\alpha_1 u^2 \pm \alpha_2 v u, \quad a \in (0, a_m), \quad x \in \Omega, \quad (1.1)$$

$$\partial_a v - \Delta_D v = -\beta_1 v^2 \pm \beta_2 u v, \quad a \in (0, a_m), \quad x \in \Omega, \quad (1.2)$$

subject to the nonlocal initial conditions

$$u(0, x) = \eta \int_0^{a_m} b_1(a) u(a, x) da, \quad x \in \Omega, \quad (1.3)$$

$$v(0, x) = \xi \int_0^{a_m} b_2(a) v(a, x) da, \quad x \in \Omega. \quad (1.4)$$

The equations are the steady-state equations of the corresponding time-dependent age-structured equations with diffusion. We refer to [30] for a recent survey on the formidable literature about age-structured population models.

The operator $-\Delta_D$ in (1.1), (1.2) stands for the negative Laplacian on Ω with subscript D indicating that Dirichlet conditions

$$u(a, x) = v(a, x) = 0, \quad a \in (0, a_m), \quad x \in \partial\Omega,$$

are imposed on the smooth boundary $\partial\Omega$ of the bounded domain Ω . Normalization to 1 of the diffusion coefficients in (1.1), (1.2) is a purely notational simplification. The number $a_m > 0$ denotes the maximal age of the species. Equations (1.3), (1.4) represent the age-boundary conditions reflecting that individuals with age zero are those created when a mother individual of any age $a \in (0, a_m)$ gives birth with rates $\eta b_1(a)$ or $\xi b_2(a)$. The functions $b_j = b_j(a) \geq 0$ describe the profiles of the fertility rates while the parameters $\eta, \xi > 0$ measure their intensity without affecting the structure of the birth profiles. For easier statements of the results we assume the birth profiles

$$b_j \in L^+_{\mathcal{O}}((0, a_m)) \text{ with } b_j(a) > 0 \text{ for } a \text{ near } a_m, \quad j = 1, 2, \quad (1.5)$$

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are normalized such that

$$\int_0^{a_m} b_j(a) e^{-\lambda_1 a} da = 1, \quad j = 1, 2, \quad (1.6)$$

where $\lambda_1 > 0$ denotes the principal eigenvalue of $-\Delta_D$ on Ω .

Assuming $\alpha_1, \alpha_2, \beta_1, \beta_2 > 0$, the form of the interaction between the two species is determined by the signs on the right hand side of equations (1.1), (1.2). Replacing \pm by a positive sign $+$ in both of the equations (1.1) and (1.2)) corresponds to a system (see (1.11), (1.12) below) where the two species are *cooperative*, while the case with \pm replaced by negative signs $-$ in each equation (1.1) and (1.2) (see (1.14), (1.15) below) reflects a *competition* of the species. The case with mixed signs, e.g. a negative sign $-$ in (1.1) instead of \pm and a positive sign $+$ in (1.2) describes a *predator-prey*-system (see (1.17), (1.18) below) with a prey density u and a predator density v .

This last case of a predator-prey-system was studied in [29] and local and global bifurcation phenomena of positive nontrivial solutions with respect to the parameters η and ξ were obtained. In the present paper, we shall derive global bifurcation results for the cooperative and the competition case. Depending on η and ξ we shall give a rather complete description of positive *coexistence solutions*, that is, of solutions (u, v) with both components u and v positive and nontrivial. Moreover, we shall also improve the local bifurcation result [29, Thm.2.4] to a global one.

We like to point out that variants of the elliptic counterparts to (1.1)-(1.2) when age-structure is neglected from the outset have been intensively studied in the past, e.g. see [4, 5, 7, 8, 9, 11, 12, 13, 18, 19, 20, 21, 22, 24, 31]. Concerning equations for a single specie, e.g. variants of (1.1) subject to (1.3), we refer to [15, 16, 25, 26, 27, 28].

To state our results for the present situation, we shall keep ξ fixed and regard η as bifurcation parameter in the following. We thus write (η, u, v) for solutions to (1.1)-(1.4) with u, v belonging to the positive cone \mathbb{W}_q^+ of

$$\mathbb{W}_q := L_q((0, a_m), W_{q,D}^2(\Omega)) \cap W_q^1((0, a_m), L_q(\Omega))$$

for $q > n + 2$ fixed, but remark that all our solutions will have smooth components u, v with respect to both $a \in J$ and $x \in \Omega$. We say that a continuum \mathfrak{C} (i.e. a closed and connected set) in $\mathbb{R}^+ \times \mathbb{W}_q^+ \times \mathbb{W}_q^+$ of solutions (η, u, v) to (1.1)-(1.4) is *unbounded with respect to η* , provided the η -projection of \mathfrak{C} contains an interval of the form (η_0, ∞) with $\eta_0 \in \mathbb{R}^+$, and we say that \mathfrak{C} is *unbounded with respect to the u -component in \mathbb{W}_q* provided there is a sequence $((\eta_j, u_j, v_j))_{j \in \mathbb{N}}$ in \mathfrak{C} with $\|u_j\|_{\mathbb{W}_q} \rightarrow \infty$ as $j \rightarrow \infty$. An analogous terminology shall be used if \mathfrak{C} is unbounded with respect to the v -component.

Clearly, problem (1.1)-(1.4) decouples when taking either u or v to be zero. Noticing that Theorem A.4 from the appendix provides for each $\eta > 1$ a unique solution $u_\eta \in \mathbb{W}_q^+ \setminus \{0\}$ to

$$\partial_a u - \Delta_D u = -\alpha_1 u^2, \quad u(0, \cdot) = \eta \int_0^{a_m} b_1(a) u(a, \cdot) da, \quad (1.7)$$

and similarly for each $\xi > 1$ a unique solution $v_\xi \in \mathbb{W}_q^+ \setminus \{0\}$ to

$$\partial_a v - \Delta_D v = -\beta_1 v^2, \quad v(0, \cdot) = \xi \int_0^{a_m} b_2(a) v(a, \cdot) da, \quad (1.8)$$

there is, independent of what the signs \pm in (1.1), (1.2) are, for any $\xi \geq 0$ the trivial branch

$$\mathfrak{B}_0 := \{(\eta, 0, 0); \eta \geq 0\}$$

and the semi-trivial branch

$$\mathfrak{B}_1 := \{(\eta, u_\eta, 0); \eta > 1\} \subset \mathbb{R}^+ \times (\mathbb{W}_q^+ \setminus \{0\}) \times \mathbb{W}_q^+ \quad (1.9)$$

of solutions. For $\xi > 1$, an additional semi-trivial branch

$$\mathfrak{B}_2 := \{(\eta, 0, v_\xi); \eta \geq 0\} \subset \mathbb{R}^+ \times \mathbb{W}_q^+ \times (\mathbb{W}_q^+ \setminus \{0\}) \quad (1.10)$$

exists. Depending on the signs \pm in (1.1), (1.2) we shall establish further local and global bifurcation of coexistence solutions from these semi-trivial branches.

1.1. Cooperative Systems. We first consider the cooperative case

$$\partial_a u - \Delta_D u = -\alpha_1 u^2 + \alpha_2 v u, \quad a \in (0, a_m), \quad x \in \Omega, \quad (1.11)$$

$$\partial_a v - \Delta_D v = -\beta_1 v^2 + \beta_2 u v, \quad a \in (0, a_m), \quad x \in \Omega, \quad (1.12)$$

subject to the nonlocal initial conditions

$$u(0, x) = \eta \int_0^{a_m} b_1(a) u(a, x) da, \quad v(0, x) = \xi \int_0^{a_m} b_2(a) v(a, x) da, \quad x \in \Omega.$$

Recall that \mathfrak{B}_1 is the only semi-trivial branch of solutions to (1.11)-(1.12) if $\xi < 1$. Thus, if $\xi < 1$ we shall derive bifurcation with respect to the parameter η from the semi-trivial branch \mathfrak{B}_1 consisting of solutions of the form $(\eta, u_\eta, 0)$. The case $\xi < 1$ is therefore more involved than the case $\xi > 1$ where we shall establish a bifurcation from the semi-trivial branch \mathfrak{B}_2 consisting of solutions of the form $(\eta, 0, v_\xi)$.

Theorem 1.1. *There is $\nu \in [0, 1)$ with the property that for each $\xi \in (\nu, 1)$ there exists $\eta_0 := \eta_0(\xi) > 1$ such that $(\eta_0, u_{\eta_0}, 0) \in \mathfrak{B}_1$ is a bifurcation point. There is an unbounded continuum \mathfrak{C}_1 of coexistence solutions (η, u, v) in $\mathbb{R}^+ \times (\mathbb{W}_q^+ \setminus \{0\}) \times (\mathbb{W}_q^+ \setminus \{0\})$ to (1.11)-(1.12) subject to (1.3)-(1.4) emanating from $(\eta_0, u_{\eta_0}, 0)$. Near the branch \mathfrak{B}_1 , the continuum \mathfrak{C}_1 is a continuous curve. There is no other bifurcation point on \mathfrak{B}_1 to positive coexistence solutions.*

The precise values of ν and of $\eta_0(\xi) > 1$ are related to spectral radii of some operators and are given in (3.2) and (3.3), respectively. Actually, we conjecture that $\nu = 0$, see Remark 3.1.

If $\xi > 1$, then a global continuum of coexistence solutions emanates from the branch \mathfrak{B}_2 :

Theorem 1.2. *Given $\xi > 1$, there is $\eta_1 := \eta_1(\xi) \in (0, 1)$ such that $(\eta_1, 0, v_\xi) \in \mathfrak{B}_2$ is a bifurcation point. An unbounded continuum \mathfrak{C}_2 of coexistence solutions (η, u, v) in $\mathbb{R}^+ \times (\mathbb{W}_q^+ \setminus \{0\}) \times (\mathbb{W}_q^+ \setminus \{0\})$ to (1.11)-(1.12) subject to (1.3)-(1.4) emanates from $(\eta_1, 0, v_\xi)$. Near the branch \mathfrak{B}_2 , the continuum \mathfrak{C}_2 is a continuous curve. There is no other bifurcation point on \mathfrak{B}_2 or on \mathfrak{B}_1 to positive coexistence solutions.*

The precise value of $\eta_1(\xi)$ is given in (4.1). We can give a more specific characterization of the global nature of the continua \mathfrak{C}_1 and \mathfrak{C}_2 :

Corollary 1.3. *The global continua \mathfrak{C}_1 and \mathfrak{C}_2 provided by Theorem 1.1 and Theorem 1.2, respectively, are unbounded with respect to both the parameter η and the u -component in \mathbb{W}_q , or with respect to the v -component in \mathbb{W}_q . If*

$$b_2 \in L_1((0, a_m), (1 - e^{-sa})^{-1} da) \quad (1.13)$$

for some $s > 0$, then they are unbounded with respect to the u -component in \mathbb{W}_q .

1.2. Competing Systems. Next we consider the case of two competing species:

$$\partial_a u - \Delta_D u = -\alpha_1 u^2 - \alpha_2 v u, \quad a \in (0, a_m), \quad x \in \Omega, \quad (1.14)$$

$$\partial_a v - \Delta_D v = -\beta_1 v^2 - \beta_2 u v, \quad a \in (0, a_m), \quad x \in \Omega, \quad (1.15)$$

subject to the initial conditions

$$u(0, x) = \eta \int_0^{a_m} b_1(a) u(a, x) da, \quad v(0, x) = \xi \int_0^{a_m} b_2(a) v(a, x) da, \quad x \in \Omega.$$

The following theorem characterizes the competition coexistence solutions.

Theorem 1.4. *If $\xi \leq 1$, then there is no coexistence solution $(\eta, u, v) \in \mathbb{R}^+ \times (\mathbb{W}_q^+ \setminus \{0\}) \times (\mathbb{W}_q^+ \setminus \{0\})$ to (1.14)-(1.15) subject to (1.3)-(1.4). If $\xi > 1$, then there is $\eta_2 := \eta_2(\xi) > 1$ such that $(\eta_2, 0, v_\xi) \in \mathfrak{B}_2$ is a bifurcation point. A continuum \mathfrak{C}_3 of positive coexistence solutions in $\mathbb{R}^+ \times (\mathbb{W}_q^+ \setminus \{0\}) \times (\mathbb{W}_q^+ \setminus \{0\})$ emanates from $(\eta_2, 0, v_\xi)$ satisfying the alternative*

(a) \mathfrak{C}_3 joins \mathfrak{B}_2 with \mathfrak{B}_1 , or

(b) \mathfrak{C}_3 is unbounded with respect to the parameter η ,

and near the bifurcation point $(\eta_2, 0, v_\xi)$, the continuum \mathfrak{C}_3 is a continuous curve. There exists some $N > 1$ such that alternative (a) occurs for each $\xi \in (1, N)$. Moreover, if

$$\beta_2 \geq \alpha_1, \quad \beta_1 \geq \alpha_2, \quad b_1 \geq b_2 \text{ on } (0, a_m), \quad (1.16)$$

then the η -projection of \mathfrak{C}_3 is contained in the interval $(1, \xi]$; in particular, alternative (a) occurs for each $\xi > 1$.

The value of $\eta_2(\xi)$ as well as the value of $\eta_3 := \eta_3(\xi)$ corresponding to the point $(\eta_3, u_{\eta_3}, 0) \in \mathfrak{B}_1$ where \mathfrak{C}_3 joins up with \mathfrak{B}_1 if alternative (a) occurs, are determined exactly, see (5.2) and (5.6). Actually, we conjecture $N = \infty$ even if (1.16) does not hold, see Remark 5.5. Observe that (1.16) implies a biological advantage of the specie with density u due to a higher birth but lower death rate.

1.3. Predator-Prey-Systems. The case of a predator-prey-system,

$$\partial_a u - \Delta_D u = -\alpha_1 u^2 - \alpha_2 v u, \quad a \in (0, a_m), \quad x \in \Omega, \quad (1.17)$$

$$\partial_a v - \Delta_D v = -\beta_1 v^2 + \beta_2 u v, \quad a \in (0, a_m), \quad x \in \Omega, \quad (1.18)$$

subject to the initial conditions

$$u(0, x) = \eta \int_0^{a_m} b_1(a) u(a, x) da, \quad v(0, x) = \xi \int_0^{a_m} b_2(a) v(a, x) da, \quad x \in \Omega,$$

was studied in detail in [29]. A quite complete description of the structure of positive solutions was provided when ξ is regarded as bifurcation parameter and $\eta > 0$ is kept fixed [29, Thm.2.2] or when η is regarded as bifurcation parameter and $\xi > 1$ is kept fixed [Thm.2.3]. However, for the case $\xi < 1$ being fixed with parameter η , a *local* bifurcation and thus a merely partial result was obtained in [29, Thm.2.4]. More precisely, provided that $\xi \in (\delta, 1)$ for some suitable $\delta \in [0, 1)$, it was shown in [29, Thm.2.4] that there are $\varepsilon_0 > 0$ and a unique point $(\eta_4, u_{\eta_4}, 0)$ with $\eta_4 := \eta_4(\xi) > 1$ on the semi-trivial branch \mathfrak{B}_1 such that a local continuous curve

$$\mathfrak{C}_4 := \{(\eta(\varepsilon), u(\varepsilon), v(\varepsilon)); 0 < \varepsilon < \varepsilon_0\} \subset \mathbb{R}^+ \times (\mathbb{W}_q^+ \setminus \{0\}) \times (\mathbb{W}_q^+ \setminus \{0\})$$

of positive coexistence solutions bifurcates to the right from $(\eta_4, u_{\eta_4}, 0)$. Actually, this result can be improved:

Theorem 1.5. *The branch \mathfrak{C}_4 extends to an unbounded continuum in $\mathbb{R}^+ \times (\mathbb{W}_q^+ \setminus \{0\}) \times (\mathbb{W}_q^+ \setminus \{0\})$ of positive coexistence solutions to (1.17)-(1.18) subject to (1.3)-(1.4). If (1.13) holds, then \mathfrak{C}_4 is unbounded with respect to the parameter η .*

The proof of this theorem is along the lines of the one of Theorem 1.1 with only minor modifications necessary. We shall thus omit it here.

The outline of the present paper is as follows: In the next section, Section 2, notations and some preliminary results are introduced. In Section 3 a detailed proof of Theorem 1.1 is provided so that the proof of Theorem 1.2 in Section 4 is basically a straightforward modification thereof and can thus be kept short. Section 5 is dedicated to the proof of Theorem 1.4. Appendix A contains some results regarding the semi-trivial branches induced by (1.7), (1.8) which are of importance for the proofs in Sections 3-5.

2. NOTATIONS AND PRELIMINARIES

Throughout we assume that Ω is a bounded and smooth domain in \mathbb{R}^n . We fix $q > n + 2$ and set, for $\kappa > 1/q$,

$$W_{q,D}^\kappa := W_{q,D}^\kappa(\Omega) := \{u \in W_q^\kappa; u = 0 \text{ on } \partial\Omega\},$$

where $W_q^\kappa := W_q^\kappa(\Omega)$ stands for the usual Sobolev-Slobodeckii spaces (for arbitrary $\kappa > 0$) and values on the boundary are interpreted in the sense of traces. Recall from [2, III.Thm.4.10.2] and the Sobolev embedding theorem that

$$\mathbb{W}_q \hookrightarrow C([0, a_m], W_{q,D}^{2-2/q}) \hookrightarrow C([0, a_m], C^1(\bar{\Omega})) \quad (2.1)$$

so that the trace $\gamma_0 u := u(0) \in W_{q,D}^{2-2/q} \hookrightarrow C^1(\bar{\Omega})$ is well-defined for $u \in \mathbb{W}_q$. Moreover, since also

$$\mathbb{W}_q \hookrightarrow W_q^1((0, a_m), L_q) \hookrightarrow C^{1-1/q}([0, a_m], L_q)$$

with $L_q := L_q(\Omega)$, the interpolation inequality in [2, I.Thm.2.11.1] yields in fact

$$\mathbb{W}_q \hookrightarrow C^{1-1/q-\vartheta}([0, a_m], W_q^{2\vartheta}), \quad 0 \leq \vartheta \leq 1 - 1/q. \quad (2.2)$$

Note that the interior, $\text{int}(W_{q,D}^{2-2/q,+})$, of the positive cone $W_{q,D}^{2-2/q,+}$ of $W_{q,D}^{2-2/q}$ is not empty. We set

$$\mathbb{L}_q := L_q((0, a_m), L_q(\Omega)) \quad \text{and} \quad \dot{\mathbb{W}}_q^+ := \mathbb{W}_q^+ \setminus \{0\}.$$

Put $J := [0, a_m]$. Given $\varrho > 0$ and $h \in C^\varrho(J, C(\bar{\Omega}))$, we let $\Pi_{[h]}(a, \sigma)$, $0 \leq \sigma \leq a \leq a_m$, denote the unique parabolic evolution operator corresponding to $-\Delta_D + h \in C^\varrho(J, \mathcal{L}(W_{q,D}^2, L_q))$, that is,

$$z(a) = \Pi_{[h]}(a, \sigma)\Phi, \quad a \in (\sigma, a_m),$$

defines the unique strong solution to

$$\partial_a z - \Delta_D z + h z = 0, \quad a \in (\sigma, a_m), \quad z(\sigma) = \Phi,$$

for any given $\sigma \in (0, a_m)$ and $\Phi \in L_q$ (see [2, II.Cor.4.4.1]). Note that the evolution operator is positive, i.e.

$$\Pi_{[h]}(a, \sigma)\Phi \in L_q^+, \quad 0 \leq \sigma \leq a \leq a_m, \quad \Phi \in L_q^+.$$

As J is a compact interval and $-\Delta_D$ has bounded imaginary powers, it follows from (2.1) and [2] (in particular, see I.Cor.1.3.2, III.Thm.4.8.7, III.Thm.4.10.10 therein) that the operator $-\Delta_D + h$ has maximal L_q -regularity, i.e. the operator

$$(\partial_a - \Delta_D + h, \gamma_0) \in \mathcal{L}(\mathbb{W}_q, \mathbb{L}_q \times W_{q,D}^{2-2/q})$$

is a toplinear isomorphism. In particular, $\Pi_{[h]}(\cdot, 0)\Phi \in \mathbb{W}_q$ for $\Phi \in W_{q,D}^{2-2/q}$. We set

$$H_{[h]} := \int_0^{a_m} b_1(a) \Pi_{[h]}(a, 0) da, \quad \hat{H}_{[h]} := \int_0^{a_m} b_2(a) \Pi_{[h]}(a, 0) da.$$

Then $H_{[h]}$ and $\hat{H}_{[h]}$ belong to $\mathcal{K}(W_{q,D}^{2-2/q})$, that is, they define compact linear operators on $W_{q,D}^{2-2/q}$, and they are strongly positive, that is, e.g.

$$H_{[h]}\Phi \in \text{int}(W_{q,D}^{2-2/q,+}), \quad \Phi \in W_{q,D}^{2-2/q,+} \setminus \{0\}. \quad (2.3)$$

The corresponding spectral radii $r(H_{[h]})$ and $r(\hat{H}_{[h]})$ can thus be characterized according to the Krein-Rutman theorem [1, Thm.3.2] (see Lemma A.3 from the appendix). In particular, the normalizations (1.6) readily imply

$$r(H_{[0]}) = r(\hat{H}_{[0]}) = 1 \quad (2.4)$$

since any positive eigenfunction of $-\Delta_D$ is an eigenfunction of $H_{[h]}$ and $\hat{H}_{[h]}$ as well. It is worthwhile to point out that (2.2) warrants an equivalent formulation of a solution $(u, v) \in \mathbb{W}_q \times \mathbb{W}_q$ to (1.1)-(1.4) as

$$u(a) = \Pi_{[\alpha_1 u \mp \alpha_2 v]}(a, 0) u(0), \quad a \in J, \quad u(0) = \eta H_{[\alpha_1 u \mp \alpha_2 v]} u(0), \quad (2.5)$$

$$v(a) = \Pi_{[\beta_1 v \mp \beta_2 u]}(a, 0) v(0), \quad a \in J, \quad v(0) = \xi \hat{H}_{[\beta_1 v \mp \beta_2 u]} v(0). \quad (2.6)$$

Observe that u, v are nonzero or nonnegative provided $u(0), v(0)$ are nonzero or nonnegative. Hence, if $(u, v) \in \mathbb{W}_q^+ \times \mathbb{W}_q^+$ solves (1.1)-(1.4), then

$$\eta r(H_{[\alpha_1 u \mp \alpha_2 v]}) = 1 \quad \text{if} \quad u(0) \in W_{q,D}^{2-2/q,+} \setminus \{0\}, \quad (2.7)$$

$$\xi r(\hat{H}_{[\beta_1 v \mp \beta_2 u]}) = 1 \quad \text{if} \quad v(0) \in W_{q,D}^{2-2/q,+} \setminus \{0\}, \quad (2.8)$$

owing to Lemma A.3. In particular, we have

$$\eta r(H_{[\alpha_1 u_\eta]}) = \xi r(\hat{H}_{[\beta_1 v_\xi]}) = 1, \quad \eta, \xi > 1 \quad (2.9)$$

by (1.7) and (1.8) since $u_\eta(0), v_\xi(0) \in W_{q,D}^{2-2/q,+} \setminus \{0\}$. We conclude this section with the following auxiliary result:

Lemma 2.1. *Given $M > 0$ there is $c(M) > 0$ such that*

$$\|u(a)\|_\infty + \|v(a)\|_\infty \leq M, \quad a \in J, \quad (2.10)$$

implies

$$\|u\|_{\mathbb{W}_q} \leq c(M)(\eta + 1), \quad \|v\|_{\mathbb{W}_q} \leq c(M)(\xi + 1)$$

for any solution $(u, v) \in \mathbb{W}_q^+ \times \mathbb{W}_q^+$ to (1.1)-(1.4) with $\eta, \xi > 0$.

Proof. If $(u, v) \in \mathbb{W}_q^+ \times \mathbb{W}_q^+$ solves (1.1)-(1.4), we derive from (1.1), (2.10), and the property of maximal L_q -regularity of $-\Delta_D$ that there is $c_0(M) > 0$ such that

$$\|u\|_{\mathbb{W}_q} \leq c \left(\|u(0)\|_{W_{q,D}^{2-2/q}} + \|-\alpha_1 u^2 \pm \alpha_2 uv\|_{L_q} \right) \leq c_0(M) \left(\|u(0)\|_{W_{q,D}^{2-2/q}} + 1 \right).$$

Writing (1.1) in the form

$$u(a) = e^{a\Delta_D} u(0) + \int_0^a e^{(a-\sigma)\Delta_D} (-\alpha_1 u(\sigma)^2 \pm \alpha_2 u(\sigma)v(\sigma)) d\sigma, \quad a \in J,$$

and using $\|e^{a\Delta_D}\|_{\mathcal{L}(L_q, W_{q,D}^{2-2/q})} \leq ca^{1/q-1}$ for $a > 0$, we obtain from (1.3) and (2.10)

$$\begin{aligned} \|u(0)\|_{W_{q,D}^{2-2/q}} &\leq \eta \|b_1\|_\infty \int_0^{a_m} \|e^{a\Delta_D}\|_{\mathcal{L}(L_q, W_{q,D}^{2-2/q})} \|u(0)\|_{L_q} da \\ &\quad + \eta \|b_1\|_\infty \int_0^{a_m} \int_0^a \|e^{(a-\sigma)\Delta_D}\|_{\mathcal{L}(L_q, W_{q,D}^{2-2/q})} (\|\alpha_1 u(\sigma)^2\|_{L_q} + \|\alpha_2 u(\sigma)v(\sigma)\|_{L_q}) d\sigma da \\ &\leq c_1(M) \eta. \end{aligned}$$

Consequently, $\|u\|_{\mathbb{W}_q} \leq c(M)(\eta + 1)$. Similarly we deduce $\|v\|_{\mathbb{W}_q} \leq c(M)(\xi + 1)$. □

3. COOPERATIVE SYSTEMS WITH $\xi < 1$: PROOF OF THEOREM 1.1

We focus our attention on (1.11)-(1.12) subject to (1.3)-(1.4) when $\xi < 1$. First we show local bifurcation of a continuous curve from \mathfrak{B}_1 by using the results of Crandall-Rabinowitz [10]. We remark that

$$(\eta \mapsto r(\hat{H}_{[-\beta_2 u_\eta]})) \in C((1, \infty), (1, \infty)) \text{ is strictly increasing,} \quad \lim_{\eta \rightarrow 1} r(\hat{H}_{[-\beta_2 u_\eta]}) = 1 \quad (3.1)$$

according to Lemma A.3, Theorem A.4, and (2.4), so

$$\nu := \frac{1}{\lim_{\eta \rightarrow \infty} r(\hat{H}_{[-\beta_2 u_\eta]})} \in [0, 1] \quad (3.2)$$

is well-defined.

Remark 3.1. As $\|u_\eta\|_\infty \rightarrow \infty$ for $\eta \rightarrow \infty$ by Theorem A.4, we conjecture $\nu = 0$ in (3.2).

For the remainder of this section we fix $\xi \in (\nu, 1)$. The observations above ensure the existence of a unique value $\eta_0 := \eta_0(\xi) > 1$ for which

$$\xi r(\hat{H}_{[-\beta_2 u_{\eta_0}]}) = 1. \quad (3.3)$$

Note then that

$$\ker(1 - \xi \hat{H}_{[-\beta_2 u_{\eta_0}]}) = \text{span}\{\Psi_0\} \quad \text{with} \quad \Psi_0 \in \text{int}(W_{q,D}^{2-2/q,+}) \quad (3.4)$$

by the Krein-Rutman theorem. With these notations we have:

Lemma 3.2. *There is a local continuous curve $\mathfrak{C}_1 \subset \mathbb{R}^+ \times \mathbb{W}_q^+ \times \mathbb{W}_q^+$ of coexistence solutions to (1.11)-(1.12) subject to (1.3)-(1.4) bifurcating from $(\eta_0, u_{\eta_0}, 0) \in \mathfrak{B}_1$, and all positive coexistence solutions near $(\eta_0, u_{\eta_0}, 0)$ lie on this curve.*

Proof. The proof is in the spirit of the one of [29, Prop.2.7]. We are linearizing (1.11)-(1.12) around $(\eta_0, u_{\eta_0}, 0) \in \mathfrak{B}_1$. For this observe that $(\eta, u, v) = (\eta, u_\eta + w, v) \in \mathbb{R} \times \mathbb{W}_q \times \mathbb{W}_q$ solves (1.11)-(1.12) subject to (1.3)-(1.4) if and only if $(\eta, w, v) \in \mathbb{R} \times \mathbb{W}_q \times \mathbb{W}_q$ solves

$$\partial_a w - \Delta_D w = -\alpha_1 w^2 - 2\alpha_1 u_\eta w + \alpha_2 v(u_\eta + w), \quad w(0) = \eta W, \quad (3.5)$$

$$\partial_a v - \Delta_D v = -\beta_1 v^2 + \beta_2 v(u_\eta + w), \quad v(0) = \xi V, \quad (3.6)$$

where we slightly abuse notation by writing

$$W := \int_0^{a_m} b_1(a) w(a) da, \quad V := \int_0^{a_m} b_2(a) v(a) da$$

when $w, v \in \mathbb{W}_q$. We shall use this notation also for other capital letters since it will always be clear from the context, which of the profiles b_1 or b_2 is meant. Using maximal L_q -regularity of $-\Delta_D$, we may introduce the operator

$$T := (\partial_a - \Delta_D, \gamma_0)^{-1} \in \mathcal{L}(\mathbb{L}_q \times W_{q,D}^{2-2/q}, \mathbb{W}_q)$$

so that the solutions to (3.5)-(3.6) are the zeros of the function

$$F(\eta, w, u) := \begin{pmatrix} w - T(-\alpha_1 w^2 - 2\alpha_1 u_\eta w + \alpha_2 v(u_\eta + w), \eta W) \\ v - T(-\beta_1 v^2 + \beta_2 v(u_\eta + w), \xi V) \end{pmatrix}.$$

Observe that

$$F \in C^2((1, \infty) \times \mathbb{W}_q \times \mathbb{W}_q, \mathbb{W}_q \times \mathbb{W}_q)$$

with partial Frechét derivatives at $(\eta, w, v) = (\eta, 0, 0)$ given by

$$F_{(w,v)}(\eta, 0, 0)(\phi, \psi) = \begin{pmatrix} \phi - T(-2\alpha_1 u_\eta \phi + \alpha_2 u_\eta \psi, \eta \Phi) \\ \psi - T(\beta_2 u_\eta \psi, \xi \Psi) \end{pmatrix}$$

and

$$F_{\eta,(w,v)}(\eta, 0, 0)(\phi, \psi) = \begin{pmatrix} -T(-2\alpha_1 u'_\eta \phi + \alpha_2 u'_\eta \psi, \Phi) \\ -T(\beta_2 u'_\eta \psi, 0) \end{pmatrix}$$

for $(\phi, \psi) \in \mathbb{W}_q \times \mathbb{W}_q$, where $u'_\eta := \frac{\partial}{\partial \eta} u_\eta$ is well defined according to Theorem A.4. We next show that the kernel of $F_{(w,v)}(\eta_0, 0, 0)$ is one-dimensional. Given $(\phi, \psi) \in \ker(F_{(w,v)}(\eta_0, 0, 0))$ we have

$$\partial_a \phi - \Delta_D \phi = -2\alpha_1 u_{\eta_0} \phi + \alpha_2 u_{\eta_0} \psi, \quad \phi(0) = \eta_0 \Phi, \quad (3.7)$$

$$\partial_a \psi - \Delta_D \psi = \beta_2 u_{\eta_0} \psi, \quad \psi(0) = \xi \Psi. \quad (3.8)$$

From (3.8) and (3.4) we conclude that $\psi = \mu \psi_*$ for some $\mu \in \mathbb{R}$ with

$$\psi_* := \Pi_{[-\beta_2 u_{\eta_0}]}(\cdot, 0) \Psi_0 \in \mathbb{W}_q.$$

Plugging this into (3.7) and observing that $1 - \eta_0 H_{[2\alpha_1 u_{\eta_0}]}$ is invertible since

$$\eta_0 r(H_{[2\alpha_1 u_{\eta_0}]}) < \eta_0 r(H_{[\alpha_1 u_{\eta_0}]}) = 1$$

by (2.9), Lemma A.3, and the positivity of u_{η_0} , we derive $\phi = \mu\phi_*$, where

$$\phi_* := \Pi_{[2\alpha_1 u_{\eta_0}]}(\cdot, 0)\Phi_0 + S\psi_* \in \mathbb{W}_q, \quad a \in J,$$

with

$$(S\psi_*)(a) := \alpha_2 \int_0^a \Pi_{[2\alpha_1 u_{\eta_0}]}(a, \sigma) (u_{\eta_0}(\sigma)\psi_*(\sigma)) d\sigma, \quad a \in J,$$

$$\Phi_0 := \eta_0(1 - \eta_0 H_{[2\alpha_1 u_{\eta_0}]})^{-1} \int_0^{a_m} b_1(a)(S\psi_*)(a) da.$$

Therefore,

$$\ker(F_{(w,v)}(\eta_0, 0, 0)) = \text{span}\{(\phi_*, \psi_*)\}.$$

As [3, Thm.1.1] and Sobolev's embedding theorem ensure the compact embedding $\mathbb{W}_q \hookrightarrow L_\infty(J, C(\bar{\Omega}))$ since $q > n + 2$, we have

$$\mathbb{W}_q \times \mathbb{W}_q \rightarrow \mathbb{L}_q, \quad (w, v) \mapsto wv \quad \text{is compact.} \quad (3.9)$$

In particular, it readily follows that the derivative of F has the form $F_{(w,v)}(\eta_0, 0, 0) = 1 - \hat{T}$ with a compact operator \hat{T} . From this we get that also the codimension of $\text{rg}(F_{(w,v)}(\eta_0, 0, 0))$ equals one. We next check the transversality condition of [10]. For, suppose that

$$F_{\eta, (w,v)}(\eta_0, 0, 0)(\phi_*, \psi_*) \in \text{rg}(F_{(w,v)}(\eta_0, 0, 0)).$$

Then there exists $v \in \mathbb{W}_q$ with

$$v - T(\beta_2 u_{\eta_0} v, \xi V) = -T(\beta_2 u'_{\eta_0} \psi_*, 0).$$

Choosing $\tau > 0$ such that $\tau\Psi_0 - v(0) \in \text{int}(W_{q,D}^{2-2/q,+})$ and putting $p := \tau\psi_* - v$, we obtain from the definition of ψ_*

$$p(a) = \Pi_{[-\beta_2 u_{\eta_0}]}(a, 0)p(0) + \int_0^a \Pi_{[-\beta_2 u_{\eta_0}]}(a, \sigma) (\beta_2 u'_{\eta_0}(\sigma)\psi_*(\sigma)) d\sigma, \quad a \in J.$$

Thus, since $p(0) = \xi P$,

$$(1 - \xi \hat{H}_{[-\beta_2 u_{\eta_0}]})p(0) = \xi \int_0^{a_m} b_2(a) \int_0^a \Pi_{[-\beta_2 u_{\eta_0}]}(a, \sigma) (\beta_2 u'_{\eta_0}(\sigma)\psi_*(\sigma)) d\sigma da.$$

However, as the right hand side belongs to $W_{q,D}^{2-2/q,+} \setminus \{0\}$ due to (1.5), Theorem A.4, and the strong positivity of the evolution operator $\Pi_{[-\beta_2 u_{\eta_0}]}(a, \sigma)$ on $W_{q,D}^{2-2/q}$ for $0 \leq \sigma < a \leq a_m$, this last equation admits no positive solution $p(0)$ according to [1, Thm.3.2] and (3.3) which clearly contradicts the fact that $p(0) = \tau\Psi_0 - v(0)$ belongs to $\text{int}(W_{q,D}^{2-2/q,+})$. Consequently,

$$F_{\eta, (w,v)}(\eta_0, 0, 0)(\phi_*, \psi_*) \notin \text{rg}(F_{(w,v)}(\eta_0, 0, 0)).$$

We are thus in a position to apply [10, Thm.1.7] and deduce the existence of some $\varepsilon_0 > 0$ and functions $\eta \in C((-\varepsilon_0, \varepsilon_0), \mathbb{R})$ and $\theta_j \in C((-\varepsilon_0, \varepsilon_0), \mathbb{W}_q)$ with $\eta(0) = \eta_0$, $\theta_j(0) = 0$ such that the nontrivial zeros of the function F close to $(\eta_0, 0, 0)$ lie on the curve

$$\{(\eta(\varepsilon), \varepsilon(\phi_*, \psi_*) + \varepsilon(\theta_1(\varepsilon), \theta_2(\varepsilon))) ; |\varepsilon| < \varepsilon_0\}.$$

By Theorem A.4,

$$\mathfrak{C}_1 := \{(\eta(\varepsilon), u_{\eta(\varepsilon)} + \varepsilon\phi_* + \varepsilon\theta_1(\varepsilon), \varepsilon\psi_* + \varepsilon\theta_2(\varepsilon)) ; 0 < \varepsilon < \varepsilon_0\}$$

is then a continuous curve of solutions to (1.11)-(1.12), (1.3)-(1.4) bifurcating from $(\eta_0, u_{\eta_0}, 0) \in \mathfrak{B}_1$. As all traces $u_{\eta_0}(0)$, $\phi_*(0) = \Phi_0$, and $\psi_*(0) = \Psi_0$ belong to $\text{int}(W_{q,D}^{2-2/q,+})$, it follows from (2.1) and the continuity of θ_j that the initial values $u(0)$ and $v(0)$ for a solution $(\eta, u, v) \in \mathfrak{C}_1$ belong to $\text{int}(W_{q,D}^{2-2/q,+})$ provided $\varepsilon_0 > 0$ is sufficiently small, whence

$$(u, v) \in \dot{\mathbb{W}}_q^+ \times \dot{\mathbb{W}}_q^+, \quad (\eta, u, v) \in \mathfrak{C}_1,$$

by (2.5), (2.6), and positivity of the corresponding evolution operators. This completes the proof of the lemma. \square

Next we show that \mathfrak{C}_1 extends to a global continuum of positive coexistence solutions by invoking Rabinowitz' global alternative [23] along with the unilateral global results of López-Gómez [20]. The main steps of the proof are the same as in the proof of [29, Thm.2.2], but we have to argue here more subtle at several points since we are deriving bifurcation with respect to the parameter η by linearizing around a point $(\eta, u_\eta, 0) \in \mathfrak{B}_1$.

Setting $u_\eta := 0$ for $\eta \leq 1$ it follows from Theorem A.4 that

$$(\eta \mapsto u_\eta) \in C(\mathbb{R}, \mathbb{W}_q^+) . \quad (3.10)$$

Hence, defining

$$\begin{aligned} Z_1[\eta] &:= (\partial_a - \Delta_D + 2\alpha_1 u_\eta, \gamma_0)^{-1} \in \mathcal{L}(\mathbb{L}_q \times W_{q,D}^{2-2/q}, \mathbb{W}_q) , \\ Z_2[\eta] &:= (\partial_a - \Delta_D - \beta_2 u_\eta, \gamma_0)^{-1} \in \mathcal{L}(\mathbb{L}_q \times W_{q,D}^{2-2/q}, \mathbb{W}_q) , \end{aligned}$$

based on maximal L_q -regularity (see Section 2), we deduce

$$(\eta \mapsto Z_j[\eta]) \in C(\mathbb{R}, \mathcal{L}(\mathbb{L}_q \times W_{q,D}^{2-2/q}, \mathbb{W}_q)) , \quad j = 1, 2 . \quad (3.11)$$

Writing again $(\eta, u, v) = (\eta, u_\eta + w, v) \in \mathbb{R} \times \mathbb{W}_q \times \mathbb{W}_q$ and recalling (3.5) and (3.6), it follows that (1.11)-(1.12) subject to (1.3)-(1.4) may be recast equivalently as

$$(w, v) - K(\eta)(w, v) + R(\eta, w, v) = 0 \quad (3.12)$$

by setting

$$K(\eta)(w, v) := \begin{pmatrix} Z_1[\eta](\alpha_2 u_\eta v, \eta W) \\ Z_2[\eta](0, \xi V) \end{pmatrix} , \quad R(\eta, w, v) := - \begin{pmatrix} Z_1[\eta](-\alpha_1 w^2 + \alpha_2 wv, 0) \\ Z_2[\eta](-\beta_1 v^2 + \beta_2 wv, 0) \end{pmatrix}$$

for $(w, v) \in \mathbb{W}_q \times \mathbb{W}_q$ still using the notation

$$W := \int_0^{a_m} b_1(a) w(a) da , \quad V := \int_0^{a_m} b_2(a) v(a) da . \quad (3.13)$$

This notation we shall use throughout the remainder of this section as no confusion seems likely. Then (3.9), (3.10), (3.11), and the compact embedding $W_q^2 \hookrightarrow W_{q,D}^{2-2/q}$ entail

$$K(\eta) \in \mathcal{K}(\mathbb{W}_q \times \mathbb{W}_q) \text{ depends continuously on } \eta \in \mathbb{R} , \quad (3.14)$$

and

$$R \in C(\mathbb{R} \times \mathbb{W}_q \times \mathbb{W}_q, \mathbb{W}_q \times \mathbb{W}_q) \text{ is compact} \quad (3.15)$$

with

$$R(\eta, w, v) = o(\|(w, v)\|_{\mathbb{W}_q \times \mathbb{W}_q}) \text{ as } \|(w, v)\|_{\mathbb{W}_q \times \mathbb{W}_q} \rightarrow 0 , \quad (3.16)$$

uniformly with respect to η in compact intervals. Moreover, we have:

Lemma 3.3. *Let $\eta \in \mathbb{R}$. If $\mu \geq 1$ is an eigenvalue of the compact operator $K(\eta)$ with eigenvector $(w, v) \in \mathbb{W}_q \times \mathbb{W}_q$, then either $\eta > 1$ and μ/ξ is an eigenvalue of $\hat{H}_{[-\beta_2 u_\eta]}$ with eigenvector $v(0) \in W_{q,D}^{2-2/q}$ or $\mu = \eta = 1$.*

Proof. Let $\mu \geq 1$ and $(w, v) \in (\mathbb{W}_q \times \mathbb{W}_q) \setminus \{(0, 0)\}$ with $K(\eta)(w, v) = \mu(w, v)$. On the one hand, if $v = 0$, then

$$\partial_a w - \Delta_D w + 2\alpha_1 u_\eta w = 0 , \quad w(0) = \frac{\eta}{\mu} W ,$$

from which

$$w(a) = \Pi_{[2\alpha_1 u_\eta]}(a, 0)w(0) , \quad a \in J , \quad w(0) = \frac{\eta}{\mu} H_{[2\alpha_1 u_\eta]} w(0) .$$

In particular, $\eta \neq 0$ and $w(0) \neq 0$ since otherwise $(w, v) = (0, 0)$, and hence

$$\mu \leq \eta r(H_{[2\alpha_1 u_\eta]}) . \quad (3.17)$$

Next, $\eta > 1$ is impossible since otherwise $u_\eta \in \dot{\mathbb{W}}_q^+$ and so

$$\frac{\eta}{\mu} r(H_{[2\alpha_1 u_\eta]}) < \eta r(H_{[\alpha_1 u_\eta]}) = 1$$

by Lemma A.3 and (2.9) contradicting (3.17). Hence $\eta \leq 1$ and thus

$$\frac{\eta}{\mu} r(H_{[2\alpha_1 u_\eta]}) = \frac{\eta}{\mu} r(H_{[0]}) = \frac{\eta}{\mu} \leq 1$$

by (2.4) what is only possible if $\mu = \eta = 1$ according to (3.17). On the other hand, if $v \neq 0$, then from

$$\partial_a v - \Delta_D v - \beta_2 u_\eta v = 0 , \quad v(0) = \frac{\xi}{\mu} V$$

it follows

$$v(a) = \Pi_{[-\beta_2 u_\eta]}(a, 0)v(0) , \quad a \in J , \quad v(0) = \frac{\xi}{\mu} \hat{H}_{[-\beta_2 u_\eta]} v(0) ,$$

and so $v(0) \neq 0$ and $\xi \neq 0$ since otherwise $v = 0$. Consequently, μ/ξ is an eigenvalue of $\hat{H}_{[-\beta_2 u_\eta]}$ with eigenvector $v(0)$. Assuming $\eta \leq 1$ we have $u_\eta = 0$ and thus $\mu/\xi \leq r(\hat{H}_{[0]}) = 1$ by (2.4) contradicting $0 < \xi < 1$ and $\mu \geq 1$. \square

As a consequence of Lemma 3.3 the set of singular values of the family $K(\eta)$ is discrete:

Corollary 3.4. *The set $\Sigma := \{\eta \in \mathbb{R}; \dim(\ker(1 - K(\eta))) \geq 1\}$ is discrete.*

Proof. Lemma 3.3 ensures

$$\Sigma \cap (1, \infty) \subset \Xi := \{\eta > 1; \dim(\ker(1 - \xi \hat{H}_{[-\beta_2 u_\eta]})) \geq 1\} .$$

Due to

$$\Pi_{[-\beta_2 u_\eta]}(\cdot, 0)\Phi = (\partial_a - \Delta_D - \beta_2 u_\eta, \gamma_0)^{-1}(0, \Phi) , \quad \Phi \in W_{q,D}^{2-2/q} , \quad \eta > 1 ,$$

it follows from the analyticity of the inversion map for linear operators and the analyticity of the map $\eta \mapsto u_\eta$ stated in Theorem A.4 that also the map $(1, \infty) \rightarrow \mathcal{K}(W_{q,D}^{2-2/q})$, $\eta \mapsto \xi \hat{H}_{[-\beta_2 u_\eta]}$ is real analytic.

Thus, since $1 - \xi \hat{H}_{[-\beta_2 u_\eta]}$ is invertible for $\eta \in (1, \eta_0)$ owing to (3.1) and (3.3), we are in a position to apply [20, Thm.4.4.4] and conclude that Ξ is discrete. If $\eta \in \Sigma$ with $\eta \leq 1$, then necessarily $\eta = 1$ by Lemma 3.3. \square

Next, we characterize the dependence on the parameter η of the fixed point index $\text{Ind}(0, K(\eta))$ of zero with respect to $K(\eta)$. Recall that $\text{Ind}(0, K(\eta)) = (-1)^{\zeta(\eta)}$, where $\zeta(\eta)$ is the sum of the algebraic multiplicities of all real eigenvalues of $K(\eta)$ greater than one, see e.g. [20, Sect.5.6].

Lemma 3.5. *The fixed point index $\text{Ind}(0, K(\eta))$ of zero with respect to $K(\eta)$ changes sign as η crosses η_0 .*

Proof. First, let $1 < \eta < \eta_0$ and suppose there is an eigenvalue $\mu > 1$ of $K(\eta)$. Let $(w, v) \in \mathbb{W}_q \times \mathbb{W}_q$ be a corresponding eigenvector. Then Lemma 3.3 yields

$$\mu \leq \xi r(\hat{H}_{[-\beta_2 u_\eta]}) . \quad (3.18)$$

Since $\eta < \eta_0$ we have $u_\eta \leq u_{\eta_0}$ owing to Theorem A.4. But then, by Lemma A.3, (3.18), and the assumption $\mu > 1$,

$$1 < \xi r(\hat{H}_{[-\beta_2 u_\eta]}) < \xi r(\hat{H}_{[-\beta_2 u_{\eta_0}]})$$

in contradiction to the definition of η_0 in (3.3). Thus there is no eigenvalue $\mu > 1$ of $K(\eta)$ if $1 < \eta < \eta_0$, consequently

$$\text{Ind}(0, K(\eta)) = 1 , \quad 1 < \eta < \eta_0 .$$

Next, it follows from [2, II.Lem.5.1.4] and Theorem A.4 that the evolution operator $\Pi_{[-\beta_2 u_\eta]}(\cdot, 0)$ depends continuously on η (actually: analytically, cf. the proof of Corollary 3.4) and hence

$$\hat{H}_{[-\beta_2 u_\eta]} \longrightarrow \hat{H}_{[-\beta_2 u_{\eta_0}]} \text{ in } \mathcal{K}(W_{q,D}^{2-2/q}) \text{ as } \eta \longrightarrow \eta_0.$$

According to [17, IV.§3.5],

$$\lambda_2(\xi \hat{H}_{[-\beta_2 u_\eta]}) \longrightarrow \lambda_2(\xi \hat{H}_{[-\beta_2 u_{\eta_0}]}) < r(\xi \hat{H}_{[-\beta_2 u_{\eta_0}]}) = 1 \text{ as } \eta \longrightarrow \eta_0,$$

with $\lambda_2(H)$ denoting the second eigenvalue of a compact operator H . Choose $\varepsilon > 0$ with

$$\lambda_2(\xi \hat{H}_{[-\beta_2 u_\eta]}) < 1 - \varepsilon, \quad \eta_0 < \eta < \eta_0 + \varepsilon. \quad (3.19)$$

Let $\eta_0 \leq \eta < \eta_0 + \varepsilon$ and let $\mu \geq 1$ be an eigenvalue of $K(\eta)$. Then $\mu \geq 1$ is an eigenvalue of $\xi \hat{H}_{[-\beta_2 u_\eta]}$ due to Lemma 3.3 and thus $\mu = r(\xi \hat{H}_{[-\beta_2 u_\eta]}) =: \mu_*$ since μ_* is the only eigenvalue in $(1 - \varepsilon, \infty)$ by (3.19). But μ_* is a simple eigenvalue of $K(\eta)$. Indeed, noticing that

$$K(\eta)(\phi, \psi) = \mu_*(\phi, \psi)$$

is equivalent to

$$\begin{aligned} \partial_a \phi - \Delta_D \phi &= -2\alpha_1 u_\eta \phi + \frac{\alpha_2}{\mu_*} u_\eta \psi, & \phi(0) &= \frac{\eta}{\mu_*} \Phi, \\ \partial_a \psi - \Delta_D \psi &= \beta_2 u_\eta \psi, & \psi(0) &= \frac{\xi}{\mu_*} \Psi, \end{aligned}$$

it follows as in the proof of Lemma 3.3 (see (3.7) and (3.8)) that

$$\ker(K(\eta) - \mu_*) = \text{span}\{(\phi_*, \psi_*)\},$$

where

$$\psi_* := Z_2[\eta](0, \Psi_1) = \Pi_{[-\beta_2 u_\eta]}(\cdot, 0) \Psi_1 \in \dot{\mathbb{W}}_q^+$$

with $\Psi_1 \in \text{int}(W_{q,D}^{2-2/q,+})$ spanning $\ker(\mu_* - \xi \hat{H}_{[-\beta_2 u_\eta]})$ and

$$\begin{aligned} \Phi_1 &:= \frac{\eta}{\mu_*} \left(1 - \frac{\eta}{\mu_*} H_{[2\alpha_1 u_\eta]}\right)^{-1} \int_0^{a_m} b_1(a)(S\psi_*)(a) da \in W_{q,D}^{2-2/q,+}, \\ \phi_* &:= \Pi_{[2\alpha_1 u_\eta]}(\cdot, 0)\Phi_1 + S\psi_* = Z_1[\eta](S\psi_*, \Phi_1) \in \mathbb{W}_q^+. \end{aligned}$$

Invertibility of $1 - \frac{\eta}{\mu_*} H_{[2\alpha_1 u_\eta]}$ is due to $\mu_* \geq 1$, (2.9), and Lemma A.3. It then merely remains to prove that μ_* is simple. For, let $(\phi_*, \psi_*) \in \text{rg}(K(\eta) - \mu_*)$. Then

$$Z_2[\eta](0, \xi V) - \mu_* v = \psi_*$$

for some $v \in \mathbb{W}_q$, that is,

$$\partial_a v - \Delta_D v - \beta_2 u_\eta v = -\frac{1}{\mu_*} (\partial_a \psi_* - \Delta_D \psi_* - \beta_2 u_\eta \psi_*) = 0, \quad v(0) = \frac{\xi}{\mu_*} V - \frac{1}{\mu_*} \Psi_1.$$

This readily implies

$$\left(1 - \frac{\xi}{\mu_*} \hat{H}_{[-\beta_2 u_\eta]}\right) v(0) = -\frac{1}{\mu_*} \Psi_1$$

so that

$$\Psi_1 \in \ker\left(1 - \frac{\xi}{\mu_*} \hat{H}_{[-\beta_2 u_\eta]}\right) \cap \text{rg}\left(1 - \frac{\xi}{\mu_*} \hat{H}_{[-\beta_2 u_\eta]}\right)$$

contradicting the fact that the intersection equals $\{0\}$ since $\mu_*/\xi = r(\hat{H}_{[-\beta_2 u_\eta]})$ is a simple eigenvalue of $\hat{H}_{[-\beta_2 u_\eta]}$. Thus $(\phi_*, \psi_*) \notin \text{rg}(K(\xi) - \mu_*)$ and μ_* is indeed a simple eigenvalue of $K(\eta)$. This ensures

$$\text{Ind}(0, K(\eta)) = -1, \quad \eta_0 \leq \eta < \eta_0 + \varepsilon,$$

and the assertion follows. \square

Recalling the definition of Ψ_0 in (3.4) and taking $\eta = \eta_0$ (and so $\mu_* = 1$), the proof of Lemma 3.5 reveals:

Corollary 3.6. $\mu_* = 1$ is a simple eigenvalue of $K(\eta_0)$. Thus

$$\mathbb{W}_q \times \mathbb{W}_q = \ker(1 - K(\eta_0)) \oplus \text{rg}(1 - K(\eta_0)), \quad \ker(1 - K(\eta_0)) = \text{span}\{(\phi_*, \psi_*)\}$$

with $\psi_* = Z_2[\eta_0](0, \Psi_0) \in \dot{\mathbb{W}}_q^+$, $\Psi_0 = \xi \Psi_* \in \text{int}(W_{q,D}^{2-2/q,+})$, and $\phi_* \in \dot{\mathbb{W}}_q^+$.

Corollary 3.4 and Lemma 3.5 warrant that we may apply Rabinowitz' global alternative [20, Cor.6.3.2] to (3.12). Hence, we obtain a continuum \mathfrak{C}'_1 of solutions (η, u, v) in $\mathbb{R} \times \mathbb{W}_q \times \mathbb{W}_q$ to (1.11)-(1.12) subject to (1.3)-(1.4) emanating from $(\eta_0, u_{\eta_0}, 0) \in \mathfrak{B}_1$. In combination with the unilateral global bifurcation result [20, Thm.6.4.3] and Corollary 3.6 we derive that \mathfrak{C}'_1 satisfies the alternatives

- (i) \mathfrak{C}'_1 is unbounded in $\mathbb{R} \times \mathbb{W}_q \times \mathbb{W}_q$, or
- (ii) there is $\eta \in \Sigma \setminus \{\eta_0\}$ with $(\eta, u_\eta, 0) \in \mathfrak{C}'_1$, or
- (iii) there is $(\eta, u_\eta + w, v) \in \mathfrak{C}'_1$ with $(w, v) \in \text{rg}(1 - K(\eta_0)) \setminus \{(0, 0)\}$.

By Lemma 3.2, \mathfrak{C}'_1 close to $(\eta_0, u_{\eta_0}, 0)$ coincides with $\mathfrak{C}_1 \subset \mathbb{R}^+ \times \dot{\mathbb{W}}_q^+ \times \dot{\mathbb{W}}_q^+$ suggesting to abuse notation by putting

$$\mathfrak{C}_1 := \mathfrak{C}'_1 \cap (\mathbb{R}^+ \times \dot{\mathbb{W}}_q^+ \times \dot{\mathbb{W}}_q^+) \neq \emptyset.$$

In fact, we have:

Lemma 3.7. \mathfrak{C}_1 is unbounded in $\mathbb{R}^+ \times \dot{\mathbb{W}}_q^+ \times \dot{\mathbb{W}}_q^+$.

Proof. Suppose \mathfrak{C}'_1 leaves $\mathbb{R}^+ \times \dot{\mathbb{W}}_q^+ \times \dot{\mathbb{W}}_q^+$ at some point $(\eta, u, v) \in \mathfrak{C}'_1$ different from $(\eta_0, u_{\eta_0}, 0)$ and let $(\eta_j, u_j, v_j) \in \mathfrak{C}_1$ such that

$$(\eta_j, u_j, v_j) \longrightarrow (\eta, u, v) \quad \text{in } \mathbb{R} \times \mathbb{W}_q \times \mathbb{W}_q.$$

Since obviously $\eta \geq 0$, $u \geq 0$, and $v \geq 0$, the only possibility is that $u = 0$ or $v = 0$. However, as the only solutions in $\mathbb{R}^+ \times \dot{\mathbb{W}}_q^+ \times \dot{\mathbb{W}}_q^+$ close to \mathfrak{B}_0 lie on \mathfrak{B}_1 , the case $(u, v) = (0, 0)$ is impossible since $v_j \neq 0$. If $u = 0$ but $v \neq 0$, then $(\eta, u, v) = (\eta, 0, v) \in \mathfrak{C}'_1$ and $v \in \dot{\mathbb{W}}_q^+$ solves

$$\partial_a v - \Delta_D v = -\beta_1 v^2, \quad v(0) = \xi V$$

with $\xi < 1$ contradicting Theorem A.4. Consequently, $v = 0$ but $u \neq 0$, thus $u \in \dot{\mathbb{W}}_q^+$ solves

$$\partial_a u - \Delta_D u = -\alpha_1 u^2, \quad u(0) = \eta U,$$

whence $\eta > 1$ and $u = u_\eta$ by Theorem A.4. Therefore, $(\eta, u, v) = (\eta, u_\eta, 0) \in \mathfrak{C}'_1$ and we may assume that $\eta_j > 1$. To demonstrate that this also leads to a contradiction, we adapt an argument of [5, Thm.3.1]. Put $z_j := (w_j, v_j)$, where $w_j := u_j - u_{\eta_j}$, and note that $z_j \rightarrow (0, 0)$ as $j \rightarrow \infty$ by the previous observation. Moreover, since $u_j, v_j \geq 0$ and $\eta_j > 1$, we obtain from

$$\partial_a u_j - \Delta_D u_j = -\alpha_1 u_j^2 + \alpha_2 v_j u_j \geq -\alpha_1 u_j^2, \quad u_j(0) = \eta_j U_j,$$

that $u_j \geq u_{\eta_j}$ by invoking Lemma A.3, whence $z_j \in \mathbb{W}_q^+ \times \mathbb{W}_q^+$. We then define

$$Q : \mathbb{R} \times \mathbb{W}_q^2 \rightarrow \mathbb{W}_q^2, \quad Q(\zeta, z) := K(\zeta)z - R(\zeta, z)$$

and observe that Q is differentiable with respect to $z \in \mathbb{W}_q^2$, $Q(\zeta, 0) = 0$ for $\zeta \in \mathbb{R}$, and $Q(\eta_j, z_j) = z_j$. The mean value theorem ensures

$$z_j - Q_z(\eta, 0)z_j = \int_0^1 [Q_z(\eta_j, sz_j)z_j - Q_z(\eta, 0)z_j] ds$$

and hence, setting $m_j := z_j / \|z_j\|_{\mathbb{W}_q^2} \in \mathbb{W}_q^+ \times \mathbb{W}_q^+$ and taking $Q_z(\eta, 0) = K(\eta)$ into account,

$$m_j - K(\eta)m_j = \int_0^1 [Q_z(\eta_j, sz_j)m_j - Q_z(\eta, 0)m_j] ds \longrightarrow 0 \quad \text{as } j \rightarrow \infty$$

by the boundedness of $(m_j)_{j \in \mathbb{N}}$, $(\eta_j, z_j) \rightarrow (\eta, 0)$, and Lebesgue's theorem. As $K(\eta)$ is compact, this readily implies the existence of $m \in \mathbb{W}_q^+ \times \mathbb{W}_q^+$ with $\|m\|_{\mathbb{W}_q^2} = 1$ and $m = K(\eta)m$. Owing to Lemma 3.3 we conclude that $1/\xi$ is an eigenvalue of $\hat{H}_{[-\beta_2 u_\eta]}$ with positive eigenvector. Hence $1 = \xi r(\hat{H}_{[-\beta_2 u_\eta]})$ due to the Krein-Rutman theorem (see Lemma A.2) yielding $\eta = \eta_0$ what is impossible since (η, u, v) then coincides with $(\eta_0, u_{\eta_0}, 0)$. Therefore, $\mathfrak{C}_1 = \mathfrak{C}'_1$ does not leave $\mathbb{R}^+ \times \mathbb{W}_q^+ \times \mathbb{W}_q^+$ except at $(\eta_0, u_{\eta_0}, 0)$.

As a consequence of the preceding observation, alternative (ii) above can be ruled out. Suppose then that alternative (iii) above occurs, i.e. let $(\eta, u_\eta + w, v) \in \mathfrak{C}'_1$ be such that

$$(0, 0) \neq (w, v) = (1 - K(\eta_0))(f, g)$$

for some $(f, g) \in \mathbb{W}_q \times \mathbb{W}_q$. To derive a contradiction we argue similarly as in the proof of Lemma 3.2. As $v \in \mathbb{W}_q^+$, we have $v(0) = \xi V \in W_{q,D}^{2-2/q,+} \setminus \{0\}$. Recall $\psi_\star(0) = \Psi_0 \in \text{int}(W_{q,D}^{2-2/q,+})$ from Corollary 3.6 so that we may choose $\tau > 0$ with

$$g(0) - v(0) + \tau \Psi_0 \in \text{int}(W_{q,D}^{2-2/q,+}) .$$

Note that

$$v = g - Z_2[\eta_0](0, \xi G) , \quad \psi_\star = Z_2[\eta_0](0, \xi \Psi_\star) , \quad p := g - v + \tau \psi_\star = Z_2[\eta_0](0, \xi(G + \tau \Psi_\star)) .$$

The last equality reads

$$\partial_a p - \Delta_D p - \beta_2 u_{\eta_0} p = 0 , \quad p(0) = \xi(G + \tau \Psi_\star) = \xi P + \xi V ,$$

from which we deduce that

$$(1 - \xi \hat{H}_{[-\beta_2 u_{\eta_0}]})p(0) = \xi V \in W_{q,D}^{2-2/q,+} \setminus \{0\}$$

with $p(0) \in \text{int}(W_{q,D}^{2-2/q,+})$ by the choice of τ . However, this equation has no positive solution owing to [1, Thm.3.2] and the definition of ξ in (3.3). This shows that alternative (iii) above is impossible as well and the only remaining possibility is that $\mathfrak{C}_1 = \mathfrak{C}'_1$ is unbounded in $\mathbb{R}^+ \times \mathbb{W}_q^+ \times \mathbb{W}_q^+$. \square

We remark that the bifurcation point $(\eta_0, u_{\eta_0}, 0)$ is unique:

Corollary 3.8. *There is no other bifurcation point on \mathfrak{B}_1 to positive coexistence solutions than $(\eta_0, u_{\eta_0}, 0)$.*

Proof. Suppose $(\eta, u_\eta, 0) \in \mathfrak{B}_1$ is a bifurcation point to positive coexistence solutions. Approximating this point by positive solutions we derive as in the proof of Lemma 3.7 that 1 is an eigenvalue of $K(\eta)$ with an eigenvector in $\mathbb{W}_q^+ \times \mathbb{W}_q^+$ so that, according to Lemma 3.3, $1/\xi$ is an eigenvalue of $\hat{H}_{[-\beta_2 u_\eta]}$ with positive eigenvector. As above, this implies $1 = \xi r(\hat{H}_{[-\beta_2 u_\eta]})$ due to the Krein-Rutman theorem (see Lemma A.2), whence $\eta = \eta_0$. \square

This completes the proof of Theorem 1.1. It remains to give a more precise characterization of the global nature of \mathfrak{C}_1 as stated in Corollary 1.3:

Corollary 3.9. *The continuum \mathfrak{C}_1 is unbounded with respect to both the parameter η and the u -component in \mathbb{W}_q , or with respect to the v -component in \mathbb{W}_q . If (1.13) holds for some $s > 0$, then \mathfrak{C}_1 is unbounded with respect to the u -component in \mathbb{W}_q .*

Proof. (i) We have $u \geq u_\eta$ for any $(\eta, u, v) \in \mathfrak{C}_1$ with $\eta > 1$ by the comparison principle of Lemma A.1 since

$$\partial_a u - \Delta_D u = -\alpha_1 u^2 + \alpha_2 v u \geq -\alpha_1 u^2 , \quad u(0) = \eta U .$$

Since $\|u_\eta(0)\|_\infty \rightarrow \infty$ as $\eta \rightarrow \infty$ according to Theorem A.4, we conclude that \mathfrak{C}_1 is unbounded with respect to η only if it is unbounded with respect to the u -component in \mathbb{W}_q .

(ii) Next suppose (1.13) and that there is $M > s/\beta_2$ such that $\|u(a)\|_\infty \leq M$, $a \in J$, for all $(\eta, u, v) \in \mathfrak{C}_1$. Noticing

$$\partial_a v - \Delta_D v = -\beta_1 v^2 + \beta_2 u v \leq -\beta_1 v^2 + \beta_2 M v , \quad a \in (0, a_m) , \quad x \in \Omega ,$$

it follows from the parabolic maximum principle [14, Thm.13.5] that $v(a) \leq f(a)$ on $\bar{\Omega}$ for $a \in J$, where

$$f(a) := m \|v(0)\|_{\infty} (\beta_1 \|v(0)\|_{\infty} (1 - e^{-ma}) + me^{-ma})^{-1}, \quad a \in J,$$

with $m := \beta_2 M > s$ satisfies

$$f'(a) = -\beta_1 f^2(a) + mf(a), \quad a \in J, \quad f(0) = \|v(0)\|_{\infty}.$$

Thus (1.4) and (1.13) imply

$$v(0) = \xi V \leq \xi \int_0^{a_m} b_2(a) f(a) da \leq \frac{\xi m}{\beta_1} \int_0^{a_m} b_2(a) (1 - e^{-sa})^{-1} da < \infty \quad \text{on } \bar{\Omega},$$

and so, owing to the definition of f , there is some $c > 0$ such that $\|v(a)\|_{\infty} \leq c$, $a \in J$ for all $(\eta, u, v) \in \mathfrak{C}_1$. Hence, \mathfrak{C}_1 is bounded with respect to the v -component by Lemma 2.1 contradicting our findings in (i). Consequently, if (1.13) holds, then \mathfrak{C}_1 is unbounded with respect to the u -component in \mathbb{W}_q . \square

4. COOPERATIVE SYSTEMS WITH $\xi > 1$: PROOF OF THEOREM 1.2

We still focus our attention on (1.11)-(1.12) subject to (1.3)-(1.4), but let now $\xi > 1$ be arbitrarily fixed for the remainder of this section and put

$$\eta_1 := \eta_1(\xi) := \frac{1}{r(H_{[-\alpha_2 v_{\xi}]})}. \quad (4.1)$$

Then $\eta_1 \in (0, 1)$ according to (2.4) and Lemma A.3. The Krein-Rutman theorem ensures

$$\ker(1 - \eta_1 H_{[-\alpha_2 v_{\xi}]}) = \text{span}\{\Phi^0\} \quad \text{with} \quad \Phi^0 \in \text{int}(W_{q,D}^{2-2/q,+}). \quad (4.2)$$

We first prove local bifurcation of a continuous curve from $(\eta_1(\xi), 0, v_{\xi}) \in \mathfrak{B}_2$ by invoking the theorem of Crandall-Rabinowitz [10]. The present situation, however, turns out to be simpler than in the previous section.

Lemma 4.1. *A local continuous curve \mathfrak{C}_2 of positive coexistence solutions to (1.11)-(1.12) subject to (1.3)-(1.4) bifurcates from $(\eta_1(\xi), 0, v_{\xi}) \in \mathfrak{B}_2$, and all positive coexistence solutions near $(\eta_1(\xi), 0, v_{\xi})$ lie on this curve.*

Proof. We proceed similar to the proof of Lemma 3.2. Writing solutions to (1.11)-(1.12) subject to (1.3)-(1.4) in the form $(\eta, u, v) = (\eta, u, v_{\xi} + w) \in \mathbb{R} \times \mathbb{W}_q \times \mathbb{W}_q$, we have

$$\partial_a u - \Delta_D u = -\alpha_1 u^2 + \alpha_2 u(v_{\xi} + w), \quad u(0) = \eta U, \quad (4.3)$$

$$\partial_a w - \Delta_D w = -\beta_1 w^2 - 2\beta_1 v_{\xi} w + \beta_2 u(v_{\xi} + w), \quad w(0) = \xi W, \quad (4.4)$$

where we agree upon the notation (and similarly for other capital letters)

$$U := \int_0^{a_m} b_1(a) u(a) da, \quad W := \int_0^{a_m} b_2(a) w(a) da.$$

Thus we are lead to examine the zeros of the function $G \in C^2((1, \infty) \times \mathbb{W}_q \times \mathbb{W}_q, \mathbb{W}_q \times \mathbb{W}_q)$ given by

$$G(\eta, u, w) := \begin{pmatrix} u - T(-\alpha_1 u^2 + \alpha_2 u(v_{\xi} + w), \eta U) \\ w - T(-\beta_1 w^2 - 2\beta_1 v_{\xi} w + \beta_2 u(v_{\xi} + w), \xi W) \end{pmatrix},$$

with T as in the proof of Lemma 3.2. For the partial Frechét derivatives at $(\eta, u, w) = (\eta, 0, 0)$ we compute

$$G_{(u,w)}(\eta, 0, 0)(\phi, \psi) = \begin{pmatrix} \phi - T(\alpha_2 \phi v_{\xi}, \eta \Phi) \\ \psi - T(-2\beta_1 v_{\xi} \psi + \beta_2 \phi v_{\xi}, \xi \Psi) \end{pmatrix}$$

and

$$G_{\eta,(u,w)}(\eta, 0, 0)(\phi, \psi) = \begin{pmatrix} -T(0, \Phi) \\ 0 \end{pmatrix}$$

for $(\phi, \psi) \in \mathbb{W}_q \times \mathbb{W}_q$. Arguments similar to the ones in the proof of Lemma 3.2 yield

$$\ker(G_{(u,w)}(\eta_1, 0, 0)) = \text{span}\{(\phi_\star, \psi_\star)\},$$

where (see (4.2) and Lemma A.3)

$$\phi_\star := \Pi_{[-\alpha_2 v_\xi]}(\cdot, 0) \Phi^0 \in \mathbb{W}_q^+, \quad (4.5)$$

and

$$\psi_\star := \Pi_{[2\beta_1 v_\xi]}(\cdot, 0) \Psi^0 + S\phi_\star \in \mathbb{W}_q, \quad \Psi^0 := \xi(1 - \xi \hat{H}_{[2\beta_1 v_\xi]})^{-1} \int_0^{a_m} b_2(a)(S\phi_\star)(a) da,$$

with

$$(S\phi_\star)(a) := \beta_2 \int_0^a \Pi_{[2\beta_1 v_\xi]}(a, \sigma) (v_\xi(\sigma) \phi_\star(\sigma)) d\sigma, \quad a \in J.$$

Observing that the derivative of G has the form $G_{(u,w)}(\eta_1, 0, 0) = 1 - \hat{T}$ with a compact operator \hat{T} (see (3.9)), we get that also the codimension of $\text{rg}(G_{(u,w)}(\eta_1, 0, 0))$ equals one. Next assume that

$$G_{\eta, (u,w)}(\eta_1, 0, 0)(\phi_\star, \psi_\star) \in \text{rg}(G_{(u,w)}(\eta_1, 0, 0))$$

and let $u \in \mathbb{W}_q$ be with

$$u - T(\alpha_2 v_\xi u, \eta_1 U) = -T(0, \Phi_\star).$$

Then

$$\partial_a u - \Delta_D u = \alpha_2 v_\xi u, \quad a \in J, \quad u(0) = \eta_1 U - \Phi_\star.$$

This readily implies

$$\Phi_\star = -(1 - \eta_1 H_{[-\alpha_2 v_\xi]})u(0) \in \text{rg}(1 - \eta_1 H_{[-\alpha_2 v_\xi]})$$

in contradiction to

$$\Phi_\star = \frac{1}{\eta_1} \Phi^0 \in \ker(1 - \eta_1 H_{[-\alpha_2 v_\xi]})$$

by (4.2) and (4.5) since $\eta_1 r(H_{[-\alpha_2 v_\xi]}) = 1$ is a simple eigenvalue of the compact operator $\eta_1 H_{[-\alpha_2 v_\xi]}$. Consequently,

$$G_{\eta, (u,w)}(\eta_1, 0, 0)(\phi_\star, \psi_\star) \notin \text{rg}(G_{(u,w)}(\eta_1, 0, 0)),$$

and we may again apply [10, Thm.1.7]. Thus, the nontrivial zeros of the function G lie on the curve

$$\{(\eta(\varepsilon), \varepsilon(\phi_\star, \psi_\star) + \varepsilon(\theta_1(\varepsilon), \theta_2(\varepsilon))) ; |\varepsilon| < \varepsilon_0\},$$

for some $\varepsilon_0 > 0$ and functions $\eta \in C((-\varepsilon_0, \varepsilon_0), \mathbb{R})$ and $\theta_j \in C((-\varepsilon_0, \varepsilon_0), \mathbb{W}_q)$ with $\eta(0) = \eta_1$, $\theta_j(0) = 0$. Thus,

$$\mathfrak{C}_2 := \{(\eta(\varepsilon), \varepsilon\phi_\star + \varepsilon\theta_1(\varepsilon), v_\xi + \varepsilon\psi_\star + \varepsilon\theta_2(\varepsilon)) ; 0 < \varepsilon < \varepsilon_0\}$$

defines a continuous curve of solutions to (1.11)-(1.12), (1.3)-(1.4) bifurcating from $(\eta_1, 0, v_\xi) \in \mathfrak{B}_2$. As $\phi_\star(0) = \Phi^0 \in \text{int}(W_{q,D}^{2-2/q,+})$ and $v_\xi(0) \in \text{int}(W_{q,D}^{2-2/q,+})$, it follows from (2.5) and (2.6) that

$$(u, v) \in \dot{\mathbb{W}}_q^+ \times \dot{\mathbb{W}}_q^+, \quad (\eta, u, v) \in \mathfrak{C}_2,$$

provided $\varepsilon_0 > 0$ is sufficiently small. This completes the proof of the lemma. \square

To prove the assertion on global bifurcation of Theorem 1.2 we invoke Rabinowitz' global alternative [23] and the unilateral global theorem [20] as in the proof of Theorem 1.1. Again, the present situation is considerably simpler than in the proof of Theorem 1.1.

Lemma 4.2. *The local curve \mathfrak{C}_2 extends to an unbounded continuum of coexistence solutions (η, u, v) in $\mathbb{R}^+ \times \dot{\mathbb{W}}_q^+ \times \dot{\mathbb{W}}_q^+$ to (1.11)-(1.12) subject to (1.3)-(1.4).*

Proof. Introducing the operators

$$\begin{aligned}\tilde{Z}_1 &:= (\partial_a - \Delta_D - \alpha_2 v_\xi, \gamma_0)^{-1} \in \mathcal{L}(\mathbb{L}_q \times W_{q,D}^{2-2/q}, \mathbb{W}_q), \\ \tilde{Z}_2 &:= (\partial_a - \Delta_D + 2\beta_1 v_\xi, \gamma_0)^{-1} \in \mathcal{L}(\mathbb{L}_q \times W_{q,D}^{2-2/q}, \mathbb{W}_q),\end{aligned}$$

we may rewrite (4.3)-(4.4) equivalently as

$$(u, w) - \tilde{K}(\eta)(u, w) + \tilde{R}(u, w) = 0 \quad (4.6)$$

by setting

$$\tilde{K}(\eta)(u, w) := \begin{pmatrix} \tilde{Z}_1(0, \eta U) \\ \tilde{Z}_2(\beta_2 u v_\xi, \xi W) \end{pmatrix}, \quad \tilde{R}(u, w) := - \begin{pmatrix} \tilde{Z}_1(\alpha_1 u^2 + \alpha_2 u w, 0) \\ \tilde{Z}_2(-\beta_1 w^2 + \beta_2 u w, 0) \end{pmatrix}$$

for $(u, w) \in \mathbb{W}_q \times \mathbb{W}_q$. It is now easy to check on the basis of the previous section that the analogues of (3.14), (3.15), (3.16), and accordingly Lemma 3.3, Corollary 3.4, Lemma 3.5, and Corollary 3.6 hold for \tilde{K} and \tilde{R} when replacing η_0 by η_1 . Consequently, we may apply the results [20, Cor.6.3.2, Lem.6.4.1, Thm.6.4.3] on unilateral global bifurcation to (4.6) and thus derive the existence of a continuum \mathfrak{C}'_2 of solutions (η, u, v) to (1.11)-(1.12) subject to (1.3)-(1.4) in $\mathbb{R} \times \mathbb{W}_q \times \mathbb{W}_q$ emanating from $(\eta_1, 0, v_\xi)$ and satisfying the alternatives

- (i) \mathfrak{C}'_2 is unbounded in $\mathbb{R} \times \mathbb{W}_q \times \mathbb{W}_q$, or
- (ii) \mathfrak{C}'_2 contains a point $(\eta, 0, v_\xi)$ with $\eta \in \{\zeta \in \mathbb{R}; \dim(\ker(1 - \tilde{K}(\zeta))) \geq 1\}$ and $\eta \neq \eta_1$, or
- (iii) \mathfrak{C}'_2 contains a point $(\eta, u, v_\xi + w)$ with $(u, w) \in \text{rg}(1 - \tilde{K}(\eta_1))$ with $(u, w) \neq (0, 0)$.

By Lemma 4.1, \mathfrak{C}'_2 coincides with \mathfrak{C}_2 near $(\eta_1, 0, v_\xi)$ suggesting to abuse notation by putting

$$\mathfrak{C}_2 := \mathfrak{C}'_2 \cap (\mathbb{R}^+ \times \dot{\mathbb{W}}_q^+ \times \dot{\mathbb{W}}_q^+) \neq \emptyset.$$

We then claim that this so defined continuum \mathfrak{C}_2 is unbounded in $\mathbb{R}^+ \times \dot{\mathbb{W}}_q^+ \times \dot{\mathbb{W}}_q^+$. Indeed, suppose \mathfrak{C}'_2 leaves $\mathbb{R}^+ \times \dot{\mathbb{W}}_q^+ \times \dot{\mathbb{W}}_q^+$ at some point $(\eta, u, v) \in \mathfrak{C}'_2$ different from $(\eta_1, 0, v_\xi)$ and let $(\eta_j, u_j, v_j) \in \mathfrak{C}_2$ such that

$$(\eta_j, u_j, v_j) \rightarrow (\eta, u, v) \quad \text{in } \mathbb{R} \times \mathbb{W}_q \times \mathbb{W}_q.$$

Clearly, $u = 0$ or $v = 0$. Observing that

$$\partial_a v_j - \Delta_D v_j = -\beta_1 v_j^2 + \beta_2 u_j v_j \geq -\beta_1 v_j^2, \quad v_j(0) = \xi V_j,$$

whence $v_j \geq v_\xi$ by Lemma A.1, we deduce $v \geq v_\xi$ and so $u = 0$ since $(\eta, u, v) \in \mathfrak{C}'_2 \setminus \mathfrak{C}_2$. Therefore, $(\eta, u, v) = (\eta, 0, v_\xi)$ by the uniqueness statement of Theorem A.4. A similar, but simpler argument as in Lemma 3.7 (see also [20, Lem.6.5.3] or [29, Lem.4.5]) then implies that $1/\eta$ is an eigenvalue of $H_{[-\alpha_2 v_\xi]}$ with a positive eigenvector, that is, $\eta = \eta_1$ by Lemma A.3 and (4.1) yielding the contradiction that (η, u, v) coincides with $(\eta_1, 0, v_\xi)$. Consequently, $\mathfrak{C}_2 = \mathfrak{C}'_2$ does not leave $\mathbb{R}^+ \times \dot{\mathbb{W}}_q^+ \times \dot{\mathbb{W}}_q^+$ except at $(\eta_1, 0, v_\xi)$, and we conclude that alternative (ii) above is impossible. An argument similar to the proof of Lemma 3.7 shows that alternative (iii) can also be ruled out. Therefore, the only remaining possibility is that $\mathfrak{C}'_2 = \mathfrak{C}_2$ is unbounded in $\mathbb{R}^+ \times \dot{\mathbb{W}}_q^+ \times \dot{\mathbb{W}}_q^+$. \square

It remains to prove that there is no other bifurcation point on the semi-trivial branches.

Corollary 4.3. *There is no other bifurcation point on \mathfrak{B}_2 or \mathfrak{B}_1 to positive coexistence solutions than $(\eta_1, 0, v_\xi) \in \mathfrak{B}_2$.*

Proof. If $(\eta, 0, v_\xi) \in \mathfrak{B}_2$ is a bifurcation point to positive coexistence solutions, then $1 = \eta r(H_{[-\alpha_2 v_\xi]})$ as in the proof of Corollary 3.8 and Lemma 3.7 by using (4.6), whence $\eta = \eta_1$.

Suppose that there is a bifurcation point $(\eta, u_\eta, 0)$ on \mathfrak{B}_1 to positive coexistence solutions. Then we deduce $1 = \xi r(\hat{H}_{[-\beta_2 u_\eta]})$ as in the proof of Corollary 3.8 and Lemma 3.7. However, this is not possible since $\xi > 1$ and $1 = r(\hat{H}_{[0]}) < r(\hat{H}_{[-\beta_2 u_\eta]})$ by (2.4) and Lemma A.3. \square

This completes the proof of Theorem 1.2. The proof of the following characterization of \mathfrak{C}_2 is the same as for Corollary 3.9:

Corollary 4.4. *The continuum \mathfrak{C}_2 is unbounded with respect to both the parameter η and the u -component in \mathbb{W}_q , or with respect to the v -component in \mathbb{W}_q . If (1.13) holds for some $s > 0$, then \mathfrak{C}_2 is unbounded with respect to the u -component in \mathbb{W}_q .*

5. COMPETING SYSTEMS: PROOF OF THEOREM 1.4

We next consider (1.14)-(1.15) subject to (1.3)-(1.4). The simplest case is $\xi \leq 1$ when no coexistence solutions exist:

Lemma 5.1. *If $\xi \leq 1$, then there is no solution (η, u, v) in $\mathbb{R}^+ \times \mathbb{W}_q^+ \times \dot{\mathbb{W}}_q^+$ to (1.14)-(1.15) subject to (1.3)-(1.4).*

Proof. Let $(\eta, u, v) \in \mathbb{R}^+ \times \mathbb{W}_q^+ \times \dot{\mathbb{W}}_q^+$ solve (1.14)-(1.15) subject to (1.3)-(1.4). Since $u \geq 0$, we have

$$\partial_a v - \Delta_D v \leq -\beta_1 v^2 \text{ on } J \times \Omega, \quad v(0) = \xi V, \quad (5.1)$$

and thus $z'(a) \leq -\lambda_1 z(a)$ for $a \in J$, where

$$z(a) := \int_{\Omega} \varphi_1 v(a) dx, \quad a \in J,$$

and φ_1 is a positive eigenfunction for the principal eigenvalue $\lambda_1 > 0$ of $-\Delta_D$. Therefore,

$$z(0) = \xi \int_0^{a_m} b_2(a) \int_{\Omega} \varphi_1 v(a) da dx \leq \xi \int_0^{a_m} b_2(a) e^{-\lambda_1 a} da z(0).$$

Actually, this inequality is strict and $z(0) > 0$ due to $v \in \dot{\mathbb{W}}_q^+$ and (5.1). So $\xi > 1$ by (1.6). \square

In the sequel, let $\xi > 1$ be arbitrarily fixed. The remainder of the proof of Theorem 1.4 is then very similar to the one of Theorem 1.2, and we give merely a brief sketch of the proof mainly pointing out the differences. Due to (1.6) and Lemma A.3 we have

$$\eta_2 := \eta_2(\xi) := \frac{1}{r(H_{[\alpha_2 v_\xi]})} \in (1, \infty). \quad (5.2)$$

We linearize around $(\eta_2, 0, v_\xi) \in \mathfrak{B}_2$ by writing solutions to (1.14)-(1.15) subject to (1.3)-(1.4) in the form $(\eta, u, v) = (\eta, u, v_\xi - w) \in \mathbb{R} \times \mathbb{W}_q \times \mathbb{W}_q$ with

$$(u, w) - \hat{K}(\eta)(u, w) + \hat{R}(u, w) = 0. \quad (5.3)$$

Hereby,

$$\hat{K}(\eta)(u, w) := \begin{pmatrix} \tilde{Z}_1(0, \eta U) \\ \tilde{Z}_2(\beta_2 u v_\xi, \xi W) \end{pmatrix}, \quad \hat{R}(u, w) := - \begin{pmatrix} \hat{Z}_1(-\alpha_1 u^2 + \alpha_2 u w, 0) \\ \hat{Z}_2(\beta_1 w^2 - \beta_2 u w, 0) \end{pmatrix}$$

for $(u, w) \in \mathbb{W}_q \times \mathbb{W}_q$ with

$$\begin{aligned} \hat{Z}_1 &:= (\partial_a - \Delta_D + \alpha_2 v_\xi, \gamma_0)^{-1} \in \mathcal{L}(\mathbb{L}_q \times W_{q,D}^{2-2/q}, \mathbb{W}_q), \\ \hat{Z}_2 &:= (\partial_a - \Delta_D + 2\beta_1 v_\xi, \gamma_0)^{-1} \in \mathcal{L}(\mathbb{L}_q \times W_{q,D}^{2-2/q}, \mathbb{W}_q). \end{aligned}$$

Exactly as in the previous sections we deduce with the aid of [10, 20]: there is a continuum \mathfrak{C}'_3 of solutions (η, u, v) to (1.14)-(1.15) subject to (1.3)-(1.4) in $\mathbb{R} \times \mathbb{W}_q \times \mathbb{W}_q$ emanating from $(\eta_2, 0, v_\xi) \in \mathfrak{B}_2$ and satisfying the alternatives

- (i) \mathfrak{C}'_3 is unbounded in $\mathbb{R} \times \mathbb{W}_q \times \mathbb{W}_q$, or
- (ii) there is $\eta \in \{\zeta \in \mathbb{R}; \dim(\ker(1 - \hat{K}(\zeta))) \geq 1\}$ with $\eta \neq \eta_2$ and $(\eta, 0, v_\xi) \in \mathfrak{C}'_3$, or
- (iii) there is $(\eta, u, v_\xi - w) \in \mathfrak{C}'_3$ with $(u, w) \in \text{rg}(1 - \hat{K}(\eta_2)) \setminus \{(0, 0)\}$.

Close to $(\eta_2, 0, v_\xi)$, the continuum \mathfrak{C}'_3 is a continuous curve in $\mathbb{R}^+ \times \dot{\mathbb{W}}_q^+ \times \dot{\mathbb{W}}_q^+$. We define

$$\mathfrak{C}_3 := \mathfrak{C}'_3 \cap (\mathbb{R}^+ \times \dot{\mathbb{W}}_q^+ \times \dot{\mathbb{W}}_q^+) \neq \emptyset,$$

and note that $\eta > 1$ for $(\eta, u, v) \in \mathfrak{C}_3$ by reproducing the proof of Lemma 5.1.

Lemma 5.2. *The continuum \mathfrak{C}_3 is either unbounded in $\mathbb{R}^+ \times \dot{\mathbb{W}}_q^+ \times \dot{\mathbb{W}}_q^+$ or joins \mathfrak{B}_2 with \mathfrak{B}_1 .*

Proof. Assume that $\mathfrak{C}_3 = \mathfrak{C}'_3$, that is, $\mathfrak{C}'_3 \subset \mathbb{R}^+ \times \dot{\mathbb{W}}_q^+ \times \dot{\mathbb{W}}_q^+$. Then alternative (ii) above is impossible while alternative (iii) can be ruled out with the same argument as in the proof of Lemma 3.7. Thus, if $\mathfrak{C}_3 = \mathfrak{C}'_3$, then it is unbounded in $\mathbb{R}^+ \times \dot{\mathbb{W}}_q^+ \times \dot{\mathbb{W}}_q^+$. Suppose that \mathfrak{C}_3 is a proper subset of \mathfrak{C}'_3 . Let $((\eta_j, u_j, v_j))_{j \in \mathbb{N}}$ be a sequence in \mathfrak{C}_3 converging toward $(\eta, u, v) \in \mathfrak{C}'_3 \setminus \mathfrak{C}_3$ with $(\eta, u, v) \neq (\eta_2, 0, v_\xi)$, so $u = 0$ or $v = 0$. As the only solutions close to \mathfrak{B}_0 lie on the curve \mathfrak{B}_1 , the case $(u, v) = (0, 0)$ is impossible. If $u = 0$ but $v \neq 0$, then $(\eta, u, v) = (\eta, 0, v)$ and so, since $\xi > 1$, $v = v_\xi$ by Theorem A.4. A similar, but simpler argument as in Lemma 3.7 (see also [20, Lem.6.5.3] or [29, Lem.4.5]) then implies that $\eta = \eta_2$. This yields the contradiction $(\eta, u, v) = (\eta_2, 0, v_\xi)$. Therefore, the only remaining possibility is that $v = 0$ but $u \neq 0$ from which $(\eta, u, v) = (\eta, u_\eta, 0) \in \mathfrak{B}_1$ according to (1.14), (1.3), and Theorem A.4. This proves the claim. \square

The next lemma implies, in particular, that if \mathfrak{C}_3 is unbounded in $\mathbb{R}^+ \times \dot{\mathbb{W}}_q^+ \times \dot{\mathbb{W}}_q^+$, then it is unbounded with respect to the parameter η :

Lemma 5.3. *Given $M > 1$ there is $c(M) > 0$ such that $\|u\|_{\mathbb{W}_q} + \|v\|_{\mathbb{W}_q} \leq c(M)$ whenever $(\eta, u, v) \in \mathfrak{C}_3$ with $\eta \leq M$.*

Proof. Let $(\eta, u, v) \in \mathfrak{C}_3$ with $\eta \leq M$. Recall $\xi, \eta > 1$ and observe

$$\partial_a v - \Delta_D v = -\beta_1 v^2 - \beta_2 uv \leq -\beta_1 v^2, \quad v(0) = \xi V,$$

whence

$$0 \leq v(a) \leq v_\xi(a) \leq \kappa \xi^2, \quad a \in J, \quad (5.4)$$

by Lemma A.1 and Theorem A.4. Similarly,

$$0 \leq u(a) \leq u_\eta(a) \leq \kappa \eta^2 \leq \kappa M^2, \quad a \in J. \quad (5.5)$$

Hence

$$\|u(a)\|_\infty + \|v(a)\|_\infty \leq c(M), \quad a \in J.$$

and we conclude with the help of Lemma 2.1. \square

To show that \mathfrak{C}_3 joins \mathfrak{B}_2 with \mathfrak{B}_1 for certain values of ξ , we require the following auxiliary result. Recall that φ_1 is the positive eigenfunction of $-\Delta_D$ corresponding to the principal eigenvalue $\lambda_1 > 0$ with $\|\varphi_1\|_\infty = 1$.

Lemma 5.4. *Set $\mu_1 := \lambda_1 + \alpha_2 \kappa \xi^2$ and $m_0 := e^{\alpha_2 \kappa \xi^2 a_m}$. Given $\eta > m_0$ let $z_\eta(a) := f_\eta(a) \varphi_1$, $a \in J$, where*

$$f_\eta(a) := \frac{\mu_1}{c_\eta \mu_1 e^{\mu_1 a} - \alpha_1}, \quad a \in J, \quad c_\eta := \frac{\alpha_1}{\mu_1} \frac{\eta - e^{-\lambda_1 a_m}}{\eta - m_0}.$$

Then z_η is increasing in $\eta \in (m_0, \infty)$ and $z_\eta \leq u$ on $J \times \Omega$ for any $(\eta, u, v) \in \mathfrak{C}_3$ with $\eta > m_0$.

Proof. Note that $f'_\eta + \mu_1 f_\eta = -\alpha_1 f_\eta^2$, whence

$$\partial_a z_\eta - \Delta_D z_\eta = -\alpha_1 z_\eta^2 - \alpha_2 \kappa \xi^2 z_\eta - F \text{ on } J \times \Omega,$$

where $F := \alpha_1 (f_\eta - z_\eta) z_\eta \geq 0$. Due to the definition of $c_\eta > \alpha_1 / \mu_1$ and (1.6), it is easily seen that $z_\eta(0) \leq \eta Z_\eta$. The comparison principle stated in Lemma A.2, (5.4), and (1.14) then yield $u \geq z_\eta$ on $J \times \Omega$ for any $(\eta, u, v) \in \mathfrak{C}_3$ with $\eta > m_0$. That z_η is increasing in $\eta \in (m_0, \infty)$ follows from $\partial_\eta f_\eta(a) > 0$ for $a \in J$. \square

We remark that (2.4), Lemma 5.4, Lemma A.3, and Theorem A.4 ensure that both maps $\eta \mapsto r(\hat{H}_{[\beta_2 z_\eta]})$ and $\eta \mapsto r(\hat{H}_{[\beta_2 u_\eta]})$ belong to $C((m_0, \infty), (0, 1))$ and are strictly decreasing with $r(\hat{H}_{[\beta_2 u_\eta]}) \leq r(\hat{H}_{[\beta_2 z_\eta]})$ for $\eta > m_0$ and $\lim_{\eta \rightarrow 0} r(\hat{H}_{[\beta_2 u_\eta]}) = 1$, hence

$$\left(\lim_{\eta \rightarrow \infty} r(\hat{H}_{[\beta_2 u_\eta]}) \right)^{-1} \geq \left(\lim_{\eta \rightarrow \infty} r(\hat{H}_{[\beta_2 z_\eta]}) \right)^{-1} =: N \in (1, \infty]$$

is well-defined.

Remark 5.5. As Lemma 5.4 ensures $\|z_\eta\|_\infty \rightarrow \infty$ for $\eta \rightarrow \infty$, we conjecture $N = \infty$.

Note that for any $\xi \in (1, N)$ there is a unique $\eta_3 := \eta_3(\xi) > 1$ such that

$$\xi r(\hat{H}_{[\beta_2 u_{\eta_3}]}) = 1. \quad (5.6)$$

In this case, the continuum \mathfrak{C}_3 connects \mathfrak{B}_2 with \mathfrak{B}_1 and the value of η_3 determines the point where \mathfrak{C}_3 joins up with \mathfrak{B}_1 :

Corollary 5.6. If $\xi \in (1, N)$, then \mathfrak{C}_3 joins up with \mathfrak{B}_1 at the point $(\eta_3, u_{\eta_3}, 0)$.

Proof. Let $\xi \in (1, N)$. If $(\eta, u, v) \in \mathfrak{C}_3$ with $\eta > m_0$, then $u \geq z_\eta$ by Lemma 5.4, hence

$$1 = r(\xi \hat{H}_{[\beta_1 v + \beta_2 u]}) \leq r(\xi \hat{H}_{[\beta_2 z_\eta]})$$

owing to Lemma A.3 and (2.8). Since the right hand side tends to $\xi/N < 1$ as $\eta \rightarrow \infty$, there must be some $M = M(\xi) > 1$ such that $\eta \leq M$ for any $(\eta, u, v) \in \mathfrak{C}_3$. Thus \mathfrak{C}_3 joins up with \mathfrak{B}_1 due to Lemma 5.2 and Lemma 5.3, say, at $(\hat{\eta}, u_{\hat{\eta}}, 0)$. To determine $\hat{\eta}$ we first recall that

$$(\eta, u, v) = (\eta, u_\eta - w, v) \in \mathbb{R} \times \mathbb{W}_q \times \mathbb{W}_q$$

solves (1.14)-(1.15) subject to (1.3)-(1.4) if and only if $(\eta, w, v) \in \mathbb{R} \times \mathbb{W}_q \times \mathbb{W}_q$ solves

$$\partial_a w - \Delta_D w = \alpha_1 w^2 - 2\alpha_1 u_\eta w + \alpha_2 u_\eta v - \alpha_2 v w, \quad w(0) = \eta W, \quad (5.7)$$

$$\partial_a v - \Delta_D v = -\beta_1 v^2 - \beta_2 v(u_\eta - w), \quad v(0) = \xi V, \quad (5.8)$$

where we put

$$W := \int_0^{a_m} b_1(a) w(a) da, \quad V := \int_0^{a_m} b_2(a) v(a) da.$$

Introducing

$$T := (\partial_a - \Delta_D, \gamma_0)^{-1} \in \mathcal{L}(\mathbb{L}_q \times W_{q,D}^{2-2/q}, \mathbb{W}_q)$$

and the operators

$$K_*(\eta)(w, v) := \begin{pmatrix} T(-2\alpha_1 u_\eta w + \alpha_2 u_\eta v, \eta W) \\ T(-\beta_1 u_\eta v, \xi V) \end{pmatrix}, \quad R_*(w, v) := - \begin{pmatrix} T(\alpha_1 w^2 - \alpha_2 v w, 0) \\ T(-\beta_1 v^2 + \beta_2 w v, 0) \end{pmatrix}$$

acting on $(w, v) \in \mathbb{W}_q \times \mathbb{W}_q$, equations (5.7), (5.8) are equivalent to

$$(w, v) - K_*(\eta)(w, v) + R_*(w, v) = 0. \quad (5.9)$$

The operators K_* and R_* possess the properties stated in (3.14)-(3.16). Now, as \mathfrak{C}_3 joins up with \mathfrak{B}_1 at $(\hat{\eta}, u_{\hat{\eta}}, 0)$, there is a sequence $((\eta_j, u_j, v_j))_j$ in \mathfrak{C}_3 converging to $(\hat{\eta}, u_{\hat{\eta}}, 0)$. Set $w_j := u_{\eta_j} - u_j$ and note that $w_j \in \mathbb{W}_q^+$ according to (5.5). As u_η depends continuously on η , formulation (5.9) and the properties of K_* and R_* readily imply (see, e.g., the proof of [20, Lem.6.5.3] or Lemma 3.7) that

$$\frac{(w_j, v_j)}{\|(w_j, v_j)\|_{\mathbb{W}_q \times \mathbb{W}_q}}$$

converges to some eigenvector $(\phi, \psi) \in \mathbb{W}_q^+ \times \mathbb{W}_q^+$ of $K_*(\hat{\eta})$ associated to the eigenvalue 1 and thus satisfying (5.7), (5.8) with $\eta = \hat{\eta}$ when higher order terms are neglected:

$$\begin{aligned} \partial_a \phi - \Delta_D \phi &= -2\alpha_1 u_{\hat{\eta}} \phi + \alpha_2 u_{\hat{\eta}} \psi, & \phi(0) &= \hat{\eta} \Phi, \\ \partial_a \psi - \Delta_D \psi &= -\beta_2 u_{\hat{\eta}} \psi, & \psi(0) &= \xi \Psi. \end{aligned}$$

Suppose $\psi = 0$. Then the first equation yields $\phi(0) = \hat{\eta} H_{[2\alpha_1 u_{\hat{\eta}}]} \phi(0)$ and thus, since $\phi(0) \in W_{q,D}^{2-2/q} \setminus \{0\}$, we obtain from Lemma A.3 and (2.9) the contradiction

$$1 \leq \hat{\eta} r(H_{[2\alpha_1 u_{\hat{\eta}}]}) < \hat{\eta} r(H_{[\alpha_1 u_{\hat{\eta}}]}) = 1.$$

Therefore, $\psi \neq 0$ and hence $\psi(0) \in W_{q,D}^{2-2/q} \setminus \{0\}$. The equation for ψ ensures $\psi(0) = \xi \hat{H}_{[\beta_2 u_{\hat{\eta}}]} \psi(0)$, whence $\xi r(\hat{H}_{[\beta_2 u_{\hat{\eta}}]}) = 1$. We conclude $\hat{\eta} = \eta_3$ according to (5.6). \square

Finally, we show that \mathfrak{C}_3 connects the two semi-trivial branches if assumption (1.16) holds.

Corollary 5.7. *Suppose (1.16) and let $\xi > 1$ be arbitrary. Then the η -projection of \mathfrak{C}_3 is contained in the interval $(1, \xi]$. In particular, \mathfrak{C}_3 joins \mathfrak{B}_2 with \mathfrak{B}_1 .*

Proof. Given $\Phi \in W_{q,D}^{2-2/q,+}$ and $u, v \in \mathbb{W}_q^+$ we have

$$\begin{aligned} (H_{[\alpha_1 u + \alpha_2 v]} - \hat{H}_{[\beta_1 v + \beta_2 u]}) \Phi &= \int_0^{a_m} (b_1(a) - b_2(a)) \Pi_{[\beta_1 v + \beta_2 u]}(a, 0) \Phi \, da \\ &\quad + \int_0^{a_m} b_1(a) (\Pi_{[\alpha_1 u + \alpha_2 v]}(a, 0) - \Pi_{[\beta_1 v + \beta_2 u]}(a, 0)) \Phi \, da. \end{aligned}$$

Since $\alpha_1 u + \alpha_2 v \leq \beta_1 v + \beta_2 u$ by (1.16), the parabolic maximum principle implies

$$(\Pi_{[\alpha_1 u + \alpha_2 v]}(a, 0) - \Pi_{[\beta_1 v + \beta_2 u]}(a, 0)) \Phi \geq 0 \text{ on } \Omega, \quad a \in J,$$

whence $H_{[\alpha_1 u + \alpha_2 v]} \geq \hat{H}_{[\beta_1 v + \beta_2 u]}$ by the above equality from which

$$r(H_{[\alpha_1 u + \alpha_2 v]}) \geq r(\hat{H}_{[\beta_1 v + \beta_2 u]})$$

due to [1, Thm.3.2(v)]. Thus, given $(\eta, u, v) \in \mathfrak{C}_3$ we have

$$1 = \xi r(\hat{H}_{[\beta_1 v + \beta_2 u]}) \leq \xi r(H_{[\alpha_1 u + \alpha_2 v]}) = \frac{\xi}{\eta}$$

by (2.8) and (2.7). So the η -projection of \mathfrak{C}_3 is contained in $(1, \xi]$. Due to Lemma 5.2 and Lemma 5.3, this in particular implies that \mathfrak{C}_3 joins \mathfrak{B}_2 with \mathfrak{B}_1 . \square

This completes the proof of Theorem 1.4.

APPENDIX A. AUXILIARY RESULTS: SEMI-TRIVIAL BRANCHES

In this appendix we collect certain results regarding the parameter-dependent equation

$$\partial_a u - \Delta_D u = -\alpha_1 u^2, \quad u(0, \cdot) = \eta \int_0^{a_m} b_1(a) u(a, \cdot) \, da. \quad (\text{A.1})$$

Most of these results have been proved in [29].

Suppose (1.5) and (1.6) in the following. We first recall a comparison principle (see [29, Lem.3.2]) for parabolic equations with nonlocal initial conditions of the form A.1, which, in particular, guarantees uniqueness of positive solutions:

Lemma A.1. *Let $\eta > 1$ and $F \in \mathbb{L}_q^+$. Suppose $u, v \in \dot{\mathbb{W}}_q^+$ satisfy either*

$$\begin{aligned} \partial_a u - \Delta_D u &= -\alpha_1 u^2 + F \quad \text{in } J \times \Omega, & u(0) &\geq \eta \int_0^{a_m} b_1(a) u(a) da, \\ \partial_a v - \Delta_D v &= -\alpha_1 v^2 \quad \text{in } J \times \Omega, & v(0) &= \eta \int_0^{a_m} b_1(a) v(a) da, \end{aligned}$$

or

$$\begin{aligned} \partial_a u - \Delta_D u &= -\alpha_1 u^2 \quad \text{in } J \times \Omega, & u(0) &= \eta \int_0^{a_m} b_1(a) u(a) da, \\ \partial_a v - \Delta_D v &= -\alpha_1 v^2 - F \quad \text{in } J \times \Omega, & v(0) &\leq \eta \int_0^{a_m} b_1(a) v(a) da. \end{aligned}$$

Then $u \geq v$.

Along the lines of the proof of [29, Lem.3.2] one may also derive the following variant:

Lemma A.2. *Let $\eta > 1$ and $F \in \mathbb{L}_q^+$. Let $R > 0$ and suppose $u, w, v \in \dot{\mathbb{W}}_q^+$ with $v \leq R$ on $J \times \Omega$ satisfy*

$$\begin{aligned} \partial_a u - \Delta_D u &= -\alpha_1 u^2 - \alpha_2 uv \quad \text{in } J \times \Omega, & u(0) &= \eta \int_0^{a_m} b_1(a) u(a) da, \\ \partial_a w - \Delta_D w &= -\alpha_1 w^2 - \alpha_2 R w - F \quad \text{in } J \times \Omega, & w(0) &\leq \eta \int_0^{a_m} b_1(a) w(a) da. \end{aligned}$$

Then $u \geq w$.

Properties of solutions to (A.1) are connected to operators of the form $H_{[h]}$ as introduced in Section 2. The next lemma is a consequence of the famous Krein-Rutman theorem [1, Thm.3.2] and gives information about the spectral radii of such operators. We refer to [29, Lem.3.1] for a proof. Actually, the proof of Lemma A.1 given in [29] is based on the next lemma.

Lemma A.3. *For $h \in C^e(J, C(\bar{\Omega}))$ with $\varrho > 0$, the operator $H_{[h]} \in \mathcal{K}(W_{q,D}^{2-2/q})$ is strongly positive, i.e. (2.3) holds. In particular, the spectral radius $r(H_{[h]}) > 0$ is a simple eigenvalue with an eigenfunction $B_{[h]}$ belonging to $\text{int}(W_{q,D}^{2-2/q,+})$. It is the only eigenvalue of $H_{[h]}$ with a positive eigenfunction. Moreover, if h and g both belong to $C^e(J, C(\bar{\Omega}))$ with $g \geq h$ but $g \not\equiv h$, then $r(H_{[g]}) < r(H_{[h]})$.*

Finally, we gather results from [29] about properties of solutions to (A.1) being fundamental for the investigation of (1.1)-(1.4). Recall that $\lambda_1 > 0$ is the principal eigenvalue of $-\Delta_D$ with positive eigenfunction φ_1 (normalized such that $\|\varphi_1\|_\infty = 1$).

Theorem A.4. *For each $\eta > 1$ there is a unique solution $u_\eta \in \mathbb{W}_q^+ \setminus \{0\}$ to equation (A.1). The mapping $(\eta \mapsto u_\eta) \in C^\infty((1, \infty), \mathbb{W}_q)$ is real analytic with $\|u_\eta\|_{\mathbb{W}_q} \rightarrow 0$ as $\eta \rightarrow 1$ and $\|u_\eta\|_{\mathbb{W}_q} \rightarrow \infty$ as $\eta \rightarrow \infty$. There is $\kappa > 0$ such that, for $\eta > 1$,*

$$\kappa \eta^2 \geq u_\eta(a) \geq \frac{\lambda_1}{\alpha_1} \frac{\eta - 1}{\eta(e^{\lambda_1 a} - 1) + 1 - e^{-\lambda_1(a_m - a)}} \varphi_1 \quad \text{on } \Omega, \quad a \in J, \quad (\text{A.2})$$

and $\frac{\partial}{\partial \eta} u_\eta(a) \in \text{int}(W_{q,D}^{2-2/q,+})$ for $a \in J$. If $\eta_1 > \eta_2$, then $u_{\eta_1} \geq u_{\eta_2}$. Finally, if $\eta \leq 1$, then (A.1) has no solution in $\mathbb{W}_q^+ \setminus \{0\}$.

The proof of this theorem is given in [29, Thm.2.1, Cor.3.3, Lem.3.6, Lem.3.7] except for the analyticity of the mapping $\eta \mapsto u_\eta$. However, this follows exactly as in the proof of [29, Thm.2.1] (see subsection 3.3 therein) by taking into account the real analyticity of the mapping

$$\Gamma : (1, \infty) \times \mathbb{W}_q \rightarrow \mathbb{L}_q \times W_{q,D}^{2-2/q}, \quad (\eta, u) \mapsto \left(\partial_a u - \Delta_D u + \alpha_1 u^2, u(0) - \eta \int_0^{a_m} b_1(a) u(a, \cdot) da \right)$$

and invoking the implicit function theorem for analytic maps, e.g. [6, Thm.4.5.4].

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